

# Exponentially Dichotomous Operators and Applications

Cornelis van der Mee

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## Operator Theory: Advances and Applications

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Vol. 182

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# **Exponentially Dichotomous Operators and Applications**

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Birkhäuser  
Basel · Boston · Berlin

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2000 Mathematical Subject Classification: 34B24, 35K50, 47A55, 47D06

Library of Congress Control Number: 2007942645

Bibliographic information published by Die Deutsche Bibliothek.  
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;  
detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>

ISBN 978-3-7643-8731-0 Birkhäuser Verlag AG, Basel - Boston - Berlin

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© 2008 Birkhäuser Verlag AG  
Basel · Boston · Berlin  
P.O. Box 133, CH-4010 Basel, Switzerland  
Part of Springer Science+Business Media  
Printed on acid-free paper produced from chlorine-free pulp. TCF $\infty$   
Printed in Germany

ISBN 978-3-7643-8731-0

e-ISBN 978-3-7643-8732-7

9 8 7 6 5 4 3 2 1

[www.birkhauser.ch](http://www.birkhauser.ch)

*Dedicated to the memory of my aunt Aadjé*

# Contents

<b>Preface</b>	xi
Applying Exponential Dichotomy	xi
Purpose, Limitations, and Readership	xiv
Acknowledgments	xv
<b>1 Exponentially Dichotomous Operators and Bisemigroups</b>	
1.1 Standard notation and semigroups	1
1.2 Exponentially dichotomous operators	5
1.3 Characterizing exponential dichotomy	7
1.4 Classes of exponential dichotomy	15
1.4.1 Analytic bisemigroups	16
1.4.2 Immediately norm continuous bisemigroups	20
1.4.3 Immediately compact bisemigroups	21
1.4.4 Hyperbolic semigroups	21
1.5 Adjoint and sun dual bisemigroups	24
<b>2 Perturbing Exponentially Dichotomous Operators</b>	
2.1 Invertibility in Banach algebras	28
2.1.1 Commutative Banach algebras	28
2.1.2 Noncommutative Banach algebras	29
2.2 Additive perturbations: Elementary results	33
2.2.1 Additive compact perturbations	34
2.2.2 Additive perturbations: Analytic bisemigroups	35
2.2.3 Additive perturbations: Immediate norm continuity	36
2.2.4 Compact perturbations: Immediate compactness	36
2.3 Additive perturbations: Generalizations	37
2.3.1 Applying tensor products	37
2.3.2 Bochner vs. Pettis integration	38
2.3.3 Bounded additive perturbations	41

**3 Abstract Cauchy Problems**

3.1	The abstract Cauchy problem . . . . .	47
3.1.1	Involving semigroup generators . . . . .	47
3.1.2	Involving exponentially dichotomous operators . . . . .	48
3.1.3	Involving analytic bisemigroups . . . . .	53
3.2	Characterizing exponential dichotomy . . . . .	54

**4 Riccati Equations and Wiener-Hopf Factorization**

4.1	Canonical factorization and perturbation . . . . .	61
4.1.1	Canonical factorization . . . . .	62
4.1.2	When perturbations lead to Bochner integrable kernels . . . .	64
4.1.3	When perturbations lead to Pettis integrable kernels . . . . .	70
4.2	Block operators and Riccati equations . . . . .	73
4.2.1	Riccati equations in complex Banach spaces . . . . .	74
4.2.2	Riccati equations in complex Hilbert spaces . . . . .	79
4.3	Approximating solutions of Riccati equations . . . . .	81

**5 Transport Equations**

5.1	Introduction . . . . .	89
5.2	Exponential dichotomy in transport theory . . . . .	93
5.2.1	Preliminary results . . . . .	94
5.2.2	Positive selfadjoint collision operators . . . . .	97
5.2.3	Identity plus compact collision operator . . . . .	98
5.3	Solving the boundary value problems . . . . .	101
5.3.1	Boundary value problems on a finite interval . . . . .	101
5.3.2	Boundary value problems on the half-line . . . . .	102
5.4	Avoiding compactness assumptions . . . . .	104
5.4.1	Bounded collision operators with closed range . . . . .	105
5.4.2	Unbounded collision operators with closed range . . . . .	108
5.4.2.a	General results . . . . .	109
5.4.2.b	Equivalence of the norms . . . . .	111

**6 Indefinite Sturm-Liouville Problems**

6.1	Introduction . . . . .	115
6.2	Applying positive operators on Krein spaces . . . . .	118
6.2.1	Positive operators on Krein spaces . . . . .	118
6.2.2	Applications . . . . .	121
6.3	Sturm-Liouville operators of second order . . . . .	125
6.3.1	Positive weight functions . . . . .	126
6.3.2	Indefinite weight functions . . . . .	128



6.4	Sturm-Liouville operators of any order . . . . .	133
6.4.1	Positive weight functions . . . . .	134
6.4.2	Indefinite weight functions . . . . .	136
<b>7</b>	<b>Noncausal Continuous Time Systems</b>	
7.1	Noncausal state linear systems . . . . .	140
7.1.1	Definitions, basic results, and adjoints . . . . .	140
7.1.2	Generating noncausal state linear systems . . . . .	142
7.2	Extended Pritchard-Salamon realizations . . . . .	148
7.2.1	Definitions and basic properties . . . . .	148
7.2.2	Duality of extended PS-realizations . . . . .	160
7.2.3	Generating extended PS-realizations . . . . .	163
7.2.4	Factorizing extended PS-realizations . . . . .	170
7.2.5	Solving the realization problem . . . . .	179
<b>8</b>	<b>Mixed-type Functional Differential Equations</b>	
8.1	The Green's function matrix . . . . .	191
8.2	Elementary spectral analysis . . . . .	193
8.3	Exponential dichotomy . . . . .	198
8.3.1	Decomposing the underlying Banach space . . . . .	199
8.3.2	Taking the inverse Laplace transform of the resolvent . . . . .	204
8.4	Delay equations and hyperbolic semigroups . . . . .	206
	<b>Bibliography</b> . . . . .	209
	<b>Index</b> . . . . .	221

# Preface

## Applying Exponential Dichotomy

Exponentially dichotomous operators are the natural evolution operators of first-order linear homogeneous differential equations in an arbitrary Banach space in which causal effects can have impact on both future and past events. When incorporated as the differential equation describing the state of a linear system, these systems are called noncausal or forward-backward or of mixed type. Exponentially dichotomous operators can be viewed as direct sums

$$S = S_+ \dot{+} S_-,$$

where  $S_+$  and  $S_-$  are the infinitesimal generators of exponentially decaying strongly continuous semigroups on a Banach space, one forward in time and the other backward in time. This means that its resolvent  $(\lambda - S)^{-1}$  exists on a vertical strip

$$C_\varepsilon = \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq \varepsilon\}$$

for some  $\varepsilon > 0$ , is bounded on  $C_\varepsilon$ , and is the Fourier transform of a so-called bisemigroup  $E(t)$ , composed of a semigroup forward in time on the first component space and minus a semigroup backward in time on the second component space. Both of these semigroups are exponentially decaying. The Cauchy problem governed by  $S$  now has the form

$$\begin{cases} u'(t) = Su(t) + f(t), & t \in \mathbb{R}, \\ u(0^+) - u(0^-) = x_0, \end{cases}$$

where both the inhomogeneous term  $f$  and the solution  $u$  are assumed Bochner integrable to guarantee the existence of a unique solution.

The author's interest in exponentially dichotomous operators has been sparked by his past involvement in four research areas, where it has been deemed convenient to employ exponentially dichotomous operators. These four areas are linear kinetic equations (including those governed by a Sturm-Liouville differential

operator with indefinite weight function), forward-backward systems of Pritchard-Salamon type, inverse scattering on the line, and algebraic Riccati equations related to so-called block operators of Hamiltonian type. Exponentially dichotomous operators have also arisen in the study of linear integral equations with semi-separable kernels and, more recently, in the study of functional differential equations of mixed type. We discuss each of these research areas briefly.

The research area most familiar to the author has been the mathematical modeling of stationary particle transport or radiative transfer in a spatially homogeneous plane parallel domain. Typically the boundary conditions describe the incoming particle density or incident radiative flux, which naturally requires distinguishing between the contributions of a forward and a backward direction. Here distance from the boundary takes the place of forward and backward time. Further, repeated single scattering events lead to a coupling between the contributions in the forward and backward directions. This has culminated in an extensive theory of abstract kinetic equations. A closely related application has been the use of exponentially dichotomous differential operators in the study of Sturm-Liouville equations with an indefinite weight function. We deal with kinetic equations in Chapter 5 and indefinite Sturm-Liouville problems in Chapter 6.

Linear integral equations of the second kind on intervals of the real line often have a so-called semi-separable integral kernel. This means that the kernel is separable, but the separation of variables depends on the sign of the difference between the independent variables. When the integral equation is of convolution or Hankel type in that it depends on either the difference or the sum of its arguments, its solutions can be obtained using a linear system of forward-backward type. Here the role of time is played by the independent variable. The basic results, where the linear noncausal system is finite-dimensional, were developed in the mid-1980s (cf. [17]). The theory has been refined to deal with a more extensive class of integral kernels, where the principal objective has been the investigation of a class of forward-backward systems with minimal emphasis on integral equations. We mention in particular forward-backward Pritchard-Salamon systems, but in principle even more general systems (such as natural generalizations of the well-posed linear systems studied in [148]) could be studied. We discuss two basic types of forward-backward systems in Chapter 7.

Block operators, i.e.,  $2 \times 2$  matrices whose entries are linear operators, constitute another area where exponentially dichotomous operators play an important role. Viewing such operators as additive perturbations of block diagonal operators, where the decomposition underlying the block structure renders the latter exponentially dichotomous, we are naturally led to additive (bounded) perturbation theory of exponentially dichotomous operators. Viewing the perturbed block operator as a Hamiltonian operator, its invariant subspace requirements naturally lead to algebraic Riccati equations. We thus have in hand a powerful tool for studying existence of its solutions and even approximation properties. We treat block operators and algebraic Riccati equations in detail in Chapter 4.

Delay equations have traditionally been a major source of exponentially dichotomous operators. The situation is rather special, because the component semigroup exponentially decaying backward in time is in fact a strongly continuous group. In other words, the exponentially dichotomous operators involved in treating delay equations are generators of hyperbolic semigroups. It has only been in recent years that there have been serious attempts to extend the theory of delay equations to equations with both positive and negative delays, the so-called functional differential equations of mixed type. In this case the exponentially dichotomous operators are no longer generators of hyperbolic semigroups. Another complicating factor is the apparent impossibility to apply perturbation theory for exponentially dichotomous operators. We have therefore decided to discuss functional differential equations of mixed type only in the final Chapter 8.

Exponentially dichotomous operators and the bisemigroups they generate have been introduced in the study of linear transport equations in  $L^p$ -spaces by the author [154], but the treatment fell far short of a formal definition of exponentially dichotomous operators and bisemigroups. Bart, Gohberg, and Kaashoek [16] have pioneered abstract exponential dichotomy by giving such a formal definition, by deriving some basic properties, and applying them to the realization problem for certain infinite-dimensional systems, subsequently called BGK realizations. Applications to linear integral equations with semi-separable kernels soon followed [17]. Ever since, bisemigroups have been applied in various contexts: linear transport theory, diffusion equations of indefinite Sturm-Liouville type, extended Pritchard-Salamon realizations, block operators and their various applications, and functional differential equations of mixed type. Bisemigroups appeared in the explicit expressions for the solutions of the inverse scattering problem for the matrix Zakharov-Shabat system on the line [5, 156], but it turned out later that bisemigroups could have been avoided and increased transparency been reached. Although some characterizations of exponential dichotomy were derived right from the dawn of its theory [16], it has been quite recent that more implementable characterizations have been derived [134, 38, 157].

The theory of exponential dichotomy as presented in this monograph and in its major input publications has been developed with almost total disregard of the theory of exponential dichotomy prevailing in the study of ordinary differential equations and functional differential equations. The latter theory consists of a plethora of applications to nonautonomous ordinary differential equations (see the bibliography of [142]) and functional differential equations [119, 84, 120] with various degrees of generality. Using the language of dynamical systems, Sacker and Sell have developed an umbrella theory of exponential dichotomy of linear evolution families, first in the finite-dimensional case [139, 140, 141] and more recently in infinite-dimensional Banach spaces [142]. At present, a theory of exponential dichotomy of linear evolution families within the tradition of the monograph by Chicone and Latushkin [44] on linear evolution families in complex Banach spaces awaits development.

## Purpose, Limitations, and Readership

The purpose of this monograph is to provide a unified treatment of exponentially dichotomous operators and to discuss its major applications in detail. In Chapter 1 we introduce exponentially dichotomous operators, discuss their spectral properties, and outline the special cases pertaining to specific types of constituent semigroups. We also characterize (special kinds of) exponentially dichotomous operators in terms of the operator-valued function having its resolvent as a Fourier transform. In Chapter 2 we address the problem of proving that, under reasonable assumptions, a bounded additive perturbation of an exponentially dichotomous operator is exponentially dichotomous itself. This requires discussing Fourier transforms of Bochner and Pettis integrable functions with values in general Banach algebras. The most general perturbation results will be obtained in a Hilbert space setting, but still elude us in general Banach spaces (unless the perturbation is small enough in the operator norm). In Chapter 3 we generalize the theory of Cauchy problems governed by the infinitesimal generator of a strongly continuous semigroup to the bisemigroup setting. Chapters 4–8 are devoted to applications of exponentially dichotomous operators to algebraic Riccati equations, transport theory, indefinite Sturm-Liouville diffusion equations, noncausal infinite-dimensional systems, and functional differential equations of mixed type.

In this monograph we limit ourselves to linear autonomous equations with exponential dichotomy. Strongly continuous semigroups will be discussed only as a portal to their bisemigroup counterpart. Thus we reduce to the bare minimum the discussion of results on bisemigroups which can be transcribed directly from semigroup theory by passing through the constituent semigroups. We refrain from discussing discrete-time counterparts of bisemigroups (such as those introduced in [12]), linear evolution families, their exponentially dichotomous generalizations, and any applications to nonautonomous differential and functional differential equations altogether. We have selected applications, where (i) bisemigroups are really the way to go (thus excluding a discussion of inverse scattering on the line and linear integral equations with semi-separable kernels), and (ii) there exists enough established knowledge to formulate an umbrella theory of an extensive family of applications.

Though we have made a strenuous effort to make the book self-contained, it still requires a nonnegligible basic knowledge of functional analysis. Some of the necessary material on closed linear operators, strongly continuous semigroups, Banach algebras, selfadjoint operators, integration of vector-valued functions, and compactness in spaces of bounded continuous functions is outlined in the first chapter of this monograph. In Subsection 2.3.2 we outline Bochner and Pettis integration, although by necessity vector-valued integrals will already appear in Chapter 1 in a rather intuitive way. We refer to various textbooks for details.

The audience we have in mind consists of researchers and graduate students interested in acquiring basic knowledge on exponentially dichotomous operators

and their major applications. Providing the material for a graduate course has not been our primary objective.

## Acknowledgments

The author has benefited from personal contacts with many friends and colleagues within the so-called IWOTA community, which consists of experts in linear operator theory and its applications who regularly attend the IWOTA (International Workshop on Operator Theory and its Applications) meetings and contribute to the Birkhäuser journal IEOT (Integral Equations and Operator Theory) and its accompanying OT book series.

The monograph was mostly written on the premises of the University of Cagliari, although some of it was conceived during visits to the Free University in Amsterdam. The University of Cagliari and especially colleagues like Sebastiano Seatzu and Giuseppe Rodriguez have greatly contributed to the relaxed atmosphere that has been so conducive to the conception and production processes. The author is especially grateful to André Ran for discussing the organization and contents of this book from its conception and making valuable comments on the preparation and correction of the manuscript.

The author also wishes to express his appreciation for the financial support received from the Italian Ministry of Education, Universities, and Research (MIUR) through the PRIN grant no. 20060175242-003, the Scientific Computation Group (GNCS) of the Italian Institute of Higher Mathematics (INdAM), and the University of Cagliari itself (through so-called ex-60% funding).

# Chapter 1

## Exponentially Dichotomous Operators and Bisemigroups

In this chapter we compile some basic concepts of functional analysis and employ them to define and give the basic results on exponentially dichotomous operators and strongly continuous bisemigroups. In particular, we represent the resolvents of exponentially dichotomous operators as two-sided Laplace transforms. We also discuss the special cases of analytic, immediately norm continuous, and immediately compact bisemigroups, cast hyperbolic semigroups in the bisemigroup framework, and introduce (sun) dual bisemigroups.

### 1.1 Standard notation and semigroups

The purpose of this section is to introduce some standard notation, to recall some terminology, and to present some well-known facts on strongly continuous semigroups.

**1. Bounded linear operators and direct sums.** Given the complex Banach spaces  $X$  and  $Y$ , the complex Banach space of all bounded linear operators from  $X$  into  $Y$  is denoted by  $\mathcal{L}(X, Y)$ . The norm of  $T \in \mathcal{L}(X, Y)$  is defined in the usual way:

$$\|T\|_{\mathcal{L}(X, Y)} \stackrel{\text{def}}{=} \sup_{0 \neq x \in X} (\|Tx\|_Y / \|x\|_X) = \sup_{\|x\|_X=1} \|Tx\|_Y. \quad (1.1)$$

If  $X = Y$ , we write  $\mathcal{L}(X)$  instead of  $\mathcal{L}(X, X)$ . The identity operator on  $X$  is written as  $I_X$ . If confusion is unlikely, we drop the subscript  $X$ . We write  $Z \dot{+} W$  for the algebraic direct sum of the vector spaces  $Z$  and  $W$ . It is a complex Banach space (with norm  $\|(z, w)\| \stackrel{\text{def}}{=} \|z\|_Z + \|w\|_W$ ) if  $Z$  and  $W$  are complex Banach

spaces, and a complex Hilbert space (with scalar product  $\langle (z_1, w_1), (z_2, w_2) \rangle \stackrel{\text{def}}{=} \langle z_1, z_2 \rangle + \langle w_1, w_2 \rangle$ ) if  $Z$  and  $W$  are complex Hilbert spaces.

**2. Closed linear operators.** Given the complex Banach spaces  $X$  and  $Y$ , we denote a linear operator  $S$  defined on a linear subspace of  $X$  and having its values in  $Y$  by  $S(X \rightarrow Y)$ . Its domain is written as  $\mathcal{D}(S)$ . Its kernel and range are defined by  $\text{Ker } S = \{x \in \mathcal{D}(S) : Sx = 0\}$  and  $\text{Im } S = \{Sx : x \in \mathcal{D}(S)\}$ , respectively. The linear operator  $S(X \rightarrow X)$  is said to be *closed* if its graph  $G(S) = \{(x, Sx) : x \in \mathcal{D}(S)\}$  is a closed linear subspace of  $X \dot{+} Y$ . By the Closed Graph Theorem, closed operators  $S(X \rightarrow Y)$  with domain  $\mathcal{D}(S) = X$  are bounded. We refer to [76, 166] for more details on closed linear operators.

We write  $S^*(Y^* \rightarrow X^*)$  for the closed operator which is the adjoint of a densely defined linear operator  $S(X \rightarrow Y)$ . If  $X$  and  $Y$  are complex Hilbert spaces, then the adjoint is defined in the usual way with respect to the scalar product.

The following useful result appears with full proof as Lemma 1 in [11].

**Proposition 1.1.** *For complex Banach spaces  $X$  and  $Y$ , let  $S(X \rightarrow Y)$  be a closed and densely defined linear operator. Suppose  $x \in X$  and  $z \in Y$  are such that  $\langle z, \phi \rangle = \langle x, S^* \phi \rangle$  for any  $\phi \in \mathcal{D}(S^*)$ . Then  $x \in \mathcal{D}(S)$  and  $Sx = z$ .*

*Proof.* Let  $G(S) = \{(w, Sw) : w \in \mathcal{D}(S)\}$  be the graph of  $S$ . Suppose  $(x, z) \notin G(S)$ . Since  $G(S)$  is closed in  $X \times Y$ , by the Hahn-Banach Theorem there exist  $\psi \in X^*$  and  $\chi \in Y^*$  such that  $(\psi, \chi)$  annihilates  $G(S)$  and maps  $(x, z)$  into a nonzero number, i.e.,  $\langle x, \psi \rangle \neq -\langle z, \chi \rangle$ . From  $\langle w, \psi \rangle = -\langle Sw, \chi \rangle$  for each  $w \in \mathcal{D}(S)$  it follows that  $\chi \in \mathcal{D}(S^*)$  and  $S^* \chi = -\psi$ . Then  $\langle x, S^* \chi \rangle \neq \langle z, \chi \rangle$  and  $\chi \in \mathcal{D}(S^*)$ , which is a contradiction. Consequently,  $(x, z) \in G(S)$ .  $\square$

**3. Spectrum.** Let  $T(X \rightarrow X)$  be a closed linear operator. Then the set of all  $\lambda \in \mathbb{C}$  for which  $\lambda I_X - T$  (or  $\lambda - T$ ) does not have an inverse belonging to  $\mathcal{L}(X)$ , is called the *spectrum*,  $\sigma(T)$ , of  $T$ ; it is a closed subset of  $\mathbb{C}$ . It is a nonempty compact subset of  $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$  whenever  $T \in \mathcal{L}(X)$ . Its complement in  $\mathbb{C}$  is called the *resolvent set*,  $\rho(T)$ , of  $T$ . The smallest  $r > 0$  such that  $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$  is called the *spectral radius*,  $r(T)$ , of  $T$ ; we have

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}.$$

An operator  $T \in \mathcal{L}(X, Y)$  is called *invertible* if there exists  $S \in \mathcal{L}(Y, X)$  such that  $ST = I_X$  and  $TS = I_Y$ . Then  $T \in \mathcal{L}(X)$  is invertible iff it is invertible as an element of the Banach algebra  $\mathcal{L}(X)$ .

**4. Banach space-valued function spaces.** Given a complex Banach space  $X$  and a compact Hausdorff space  $M$ , by  $C(M; X)$  we denote the complex Banach space of all continuous functions  $f : M \rightarrow X$  endowed with the norm

$$\|f\| \stackrel{\text{def}}{=} \max_{t \in M} \|f(t)\|_X. \quad (1.2)$$



If  $M$  is an arbitrary Tychonoff space instead, we denote by  $BC(M; X)$  the complex Banach space of all bounded and continuous functions  $f : M \rightarrow X$  endowed with the norm

$$\|f\| \stackrel{\text{def}}{=} \sup_{t \in M} \|f(t)\|_X. \quad (1.3)$$

If  $(E, \mu)$  is a measure space,  $X$  a complex Banach space, and  $1 \leq p \leq \infty$ , we denote by  $L^p(E; X)$  the complex Banach space of all strongly  $\mu$ -measurable functions  $f : E \rightarrow X$  for which  $\|f(\cdot)\|_X : E \rightarrow \mathbb{R}$  belongs to  $L^p(E, d\mu)$ . The norm of  $f \in L^p(E; X)$  is defined as follows:

$$\|f\| \stackrel{\text{def}}{=} \begin{cases} [\int_E (\|f(t)\|_X)^p d\mu(t)]^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{t \in E} \|f(t)\|_X, & p = \infty. \end{cases}$$

In Subsection 2.3.2 we shall discuss strong measurability and Bochner integration in more detail; see also [57, 86].

We often need the following lemma about moving a closed linear operator under the integral sign ([57, Theorem III.35]; [86, Theorem 3.7.12]). In Chapter 2 we shall prove its generalization Lemma 2.9.

**Lemma 1.2 (Hille).** *Let  $X$  and  $Y$  be complex Banach spaces and suppose  $S(X \rightarrow Y)$  is a closed linear operator. Suppose  $(E, \mu)$  is a measure space and  $f \in L^1(E; X)$ . If  $f(t) \in \mathcal{D}(S)$  for  $\mu$ -a.e.  $t \in E$  and  $Sf \in L^1(E; Y)$ , then  $\int_E f(t) d\mu(t) \in \mathcal{D}(S)$  and*

$$S \int_E f(t) d\mu(t) = \int_E Sf(t) d\mu(t).$$

**5. Semigroups.** By a (*strongly continuous*) *semigroup* on a complex Banach space  $X$  we mean a function  $E : [0, \infty) \rightarrow \mathcal{L}(X)$  such that (i)  $E(t+s) = E(t)E(s)$  for  $t, s \geq 0$ , (ii)  $E(\cdot)x : [0, \infty) \rightarrow X$  is continuous for every  $x \in X$ , and (iii)  $E(0) = I_X$ . Then the linear operator  $S(X \rightarrow X)$  defined by

$$\mathcal{D}(S) = \left\{ x \in X : \exists y \in X \text{ such that } \lim_{t \rightarrow 0^+} \left\| \frac{E(t)x - x}{t} - y \right\|_X = 0 \right\},$$

$$Sx = y,$$

is closed and densely defined and is called the (*infinitesimal*) *generator* of the semigroup  $E$ . Often one writes  $E(t) = e^{tS}$ . Since each strongly continuous semigroup has an exponential growth bound of the type  $\|E(t)\|_{\mathcal{L}(X)} = O(e^{\omega t})$  as  $t \rightarrow +\infty$  for some  $\omega \in \mathbb{R}$ , the half-plane  $\{\lambda \in \mathbb{C} : \text{Re } \lambda > \omega\}$  is contained in the resolvent set of  $S$  and we have the Laplace transform formula

$$(\lambda - S)^{-1}x = \int_0^\infty e^{-\lambda t} E(t)x dt, \quad \text{Re } \lambda > \omega, \quad x \in X, \quad (1.4)$$

where the (Bochner) integral converges absolutely in the norm of  $X$ . The infimum of all  $\omega \in \mathbb{R}$  for which  $\|E(t)\| = O(e^{\omega t})$  as  $t \rightarrow +\infty$ , is called the *exponential growth bound*,  $\omega(E)$ , of  $E$ .

Infinitesimal generators of strongly continuous semigroups can be characterized as follows [86, 166, 127, 60].

**Theorem 1.3 (Hille-Yosida-Phillips).** *A closed and densely defined linear operator  $S(X \rightarrow X)$  is the infinitesimal generator of a strongly continuous semigroup with exponential growth bound  $\omega$  if and only if*

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega\} \subset \rho(S)$$

*and there exists a constant  $M$  such that*

$$\|(\lambda - S)^{-n}\| \leq \frac{M}{(\operatorname{Re} \lambda - \omega)^n}, \quad n = 1, 2, 3, \dots$$

For later use we also present the following result [53, 54] (also [127, Theorem IV4.1] and [8, Theorem 5.1.2]).

**Theorem 1.4 (Datko).** *Let  $E : [0, \infty) \rightarrow \mathcal{L}(X)$  be a strongly continuous semigroup on a complex Banach space  $X$  such that*

$$\int_0^\infty \|E(t)x\|_X dt < \infty, \quad x \in X.$$

*Then there exist positive constants  $M$  and  $\mu$  such that*

$$\|E(t)\|_{\mathcal{L}(X)} \leq M e^{-\mu t}, \quad t \in \mathbb{R}^+.$$

*In other words,  $E$  has a negative exponential growth bound.*

**6. Ascoli-Arzelà theorem.** Let  $\mathcal{F}$  be a family of functions from a topological space  $X$  into a metric space  $(Y, \sigma)$ . Then  $\mathcal{F}$  is called *equicontinuous* at the point  $x \in X$  if for every  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $x$  such that  $\sigma(f(x), f(y)) < \varepsilon$  for each  $y \in U$  and each  $f \in \mathcal{F}$ . The family  $\mathcal{F}$  is called *equicontinuous* if it is equicontinuous at each point  $x \in X$ . In particular, each  $f \in \mathcal{F}$  is a continuous function from  $X$  into  $Y$ .

We have [136, Theorem 40]

**Theorem 1.5 (Ascoli-Arzelà).** *Let  $\mathcal{F}$  be an equicontinuous family of functions from a separable topological space  $X$  into a metric space  $Y$ . Let  $\{f_n\}_{n=1}^\infty$  be a sequence in  $\mathcal{F}$  such that for each  $x \in X$  the closure of the set  $\{f_n(x) : n \in \mathbb{N}\}$  in  $Y$  is compact. Then there is a subsequence  $\{f_{n_k}\}_{k=1}^\infty$  that converges pointwise to a continuous function  $f : X \rightarrow Y$ , and the convergence is uniform on compact subsets of  $X$ .*

In Chapter 8 we shall frequently apply Theorem 1.5 for  $X = \mathbb{R}$  (or  $X = \mathbb{R}^\pm$ ) and  $Y = \mathbb{C}^M$ . The most general versions of Theorem 1.5 can be found in [104, Ch. 7].

## 1.2 Exponentially dichotomous operators

Strongly continuous bisemigroups and exponentially dichotomous operators were formally introduced by Bart, Gohberg, and Kaashoek [16] to provide state space representations of kernels and solutions of (systems of) integral equations on the half-line and on finite intervals and to represent operator functions analytic on a strip about the real line as transfer functions of infinite-dimensional linear systems. Here we restrict ourselves to giving definitions and basic properties of exponentially dichotomous operators.

Let  $X$  be a complex Banach space. By a (*strongly continuous*) *bisemigroup* we mean a function  $E : \mathbb{R} \setminus \{0\} \rightarrow \mathcal{L}(X)$  having the following properties:

- (i) For  $t, s > 0$  we have  $E(t)E(s) = E(t+s)$  and for  $t, s < 0$  we have  $E(t)E(s) = -E(t+s)$ .
- (ii) For every  $x \in X$  the function  $E(\cdot)x : \mathbb{R} \rightarrow X$  is continuous, apart from a jump discontinuity in  $t = 0$ . That is,

$$\lim_{t \rightarrow 0^\pm} \|E(t)x - E(0^\pm)x\|_X = 0, \quad x \in X.$$

- (iii)  $E(0^+)x - E(0^-)x = x$  for every  $x \in X$ .

- (iv) There exist  $M, r > 0$  such that  $\|E(t)\| \leq Me^{-r|t|}$  for  $0 \neq t \in \mathbb{R}$ .

Any strongly continuous semigroup  $E : [0, \infty) \rightarrow X$  having a negative exponential growth bound extends to a strongly continuous bisemigroup when defining  $E(t) = 0_{\mathcal{L}(X)}$  for  $t < 0$ .

Clearly, the above properties (i) and (iii) imply that  $E(0^+)$  and  $-E(0^-)$  are bounded complementary projections on  $X$ . We call  $P = -E(0^-)$  the *separating projection* of the bisemigroup  $E$ . We obviously have

$$\begin{cases} E(t)[\text{Ker } P] \subset \text{Ker } P, & t > 0, \\ E(t)[\text{Im } P] \subset \text{Im } P, & t < 0. \end{cases} \quad (1.5)$$

The restriction of  $E(t)$  to  $\text{Ker } P$  is a strongly continuous semigroup on  $\text{Ker } P$ , while the restriction of  $-E(-t)$  to  $\text{Im } P$  is a strongly continuous semigroup on  $\text{Im } P$ . These two semigroups are called the *constituent semigroups* of the bisemigroup  $E$ . Conversely, starting from the strongly continuous semigroups  $E_j : [0, \infty) \rightarrow X_j$  ( $j = 1, 2$ ), both having a negative exponential growth bound, we can define the strongly continuous bisemigroup  $E$  on  $X = X_1 \dot{+} X_2$  by

$$E(t) = \begin{cases} E_1(t) \dot{+} 0_{X_2}, & t > 0, \\ 0_{X_1} \dot{+} (-E_2(-t)), & t < 0, \end{cases}$$

which has  $E_1$  and  $E_2$  as its constituent semigroups. By the *pair of exponential growth bounds* of a bisemigroup  $E$ , we mean the pair of (necessarily negative)

exponential growth bounds of its constituent semigroups:

$$\{\omega_+(E), \omega_-(E)\}.$$

By its *exponential growth bound*,  $\omega(E)$ , we mean the number

$$\omega(E) \stackrel{\text{def}}{=} \max(\omega_+(E), \omega_-(E)) < 0.$$

Let  $S_+(\text{Ker } P \rightarrow \text{Ker } P)$  and  $-S_-(\text{Im } P \rightarrow \text{Im } P)$  stand for the infinitesimal generators of the constituent semigroups of the bisemigroup  $E$  on  $X$ . Then the linear operator  $S(X \rightarrow X)$  defined by

$$\begin{aligned} \mathcal{D}(S) &= \{x_+ + x_- : x_+ \in \mathcal{D}(S_+), x_- \in \mathcal{D}(S_-)\}, \\ S(x_+ + x_-) &= (S_+x_+) - (S_-x_-), \end{aligned}$$

is called the (*infinitesimal*) *generator* of the bisemigroup  $E$ . Obviously,  $S(X \rightarrow X)$  is closed and densely defined. The Laplace transform formulas

$$\begin{aligned} (\lambda - S_+)^{-1}x_+ &= \int_0^\infty e^{-\lambda t} E(t)x_+ dt, \quad x_+ \in \text{Ker } P, \quad \text{Re } \lambda > \omega_+(E), \\ (-\lambda + S_-)^{-1}x_- &= - \int_0^\infty e^{\lambda t} E(-t)x_- dt, \quad x_- \in \text{Im } P, \quad \text{Re } (-\lambda) > \omega_-(E), \end{aligned}$$

where both of  $\omega_\pm(E) < 0$ , then imply the Laplace transform formula

$$(\lambda - S)^{-1}x = \int_{-\infty}^\infty e^{-\lambda t} E(t)x dt, \quad \omega_+(E) < \text{Re } \lambda < -\omega_-(E), \quad (1.6)$$

where the (Bochner) integral converges absolutely in the norm of  $X$ . Thus there exists a vertical strip in the complex plane about the imaginary axis contained in the resolvent set of the infinitesimal generator  $S$  of  $E$ . From now on we write  $E(t; S)$  for the strongly continuous bisemigroup with infinitesimal generator  $S$ .

A closed and densely defined linear operator  $S(X \rightarrow X)$  on a complex Banach space  $X$  is called *exponentially dichotomous* if it is the infinitesimal generator of a strongly continuous bisemigroup  $E$  on  $X$ . For any  $x \in X$  and  $x^* \in X^*$  the weak form of the Laplace transform formula, viz.

$$\langle (\lambda - S)^{-1}x, x^* \rangle = \int_{-\infty}^\infty e^{-\lambda t} \langle E(t)x, x^* \rangle dt, \quad \omega_+(E) < \text{Re } \lambda < -\omega_-(E), \quad (1.7)$$

allows one to reconstruct  $\langle E(t)x, x^* \rangle$  uniquely from  $S$ , simply by applying the Laplace inversion formula [164]. As a result, there exists only one bisemigroup  $E$  having  $S$  as its infinitesimal generator.

### 1.3 Characterizing exponential dichotomy

Starting from a closed and densely defined linear operator  $S$  on a complex Banach space, we now seek necessary and sufficient conditions in order that  $S$  is exponentially dichotomous. In contrast with the characterization of infinitesimal generators of strongly continuous semigroups, this problem does not have a straightforward solution that is easily applied to concrete examples. The main problem is the construction of the separating projection of the bisemigroup generated by  $S$ . Once the infinitesimal generators of the constituent semigroups are known, the above characterization problem reduces to standard semigroup theory. Our objective is to find necessary and sufficient conditions for  $S$  to be exponentially dichotomous without knowing the separating projection in advance. There are two types of characterization: one in terms of the resolvent of  $S$  and one in terms of the inverse Laplace transform  $E(t)$  of  $(\lambda - S)^{-1}$ .

To be exponentially dichotomous, a closed and densely defined linear operator  $S(X \rightarrow X)$  should **at least** have the following two properties:

- (a) There exists  $h > 0$  such that

$$C_h \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} : -h \leq \operatorname{Re} \lambda \leq h\} \subset \rho(S).$$

- (b)  $(\lambda - S)^{-1}$  is bounded in  $\lambda \in C_h$ .

These two conditions will henceforth be called *conditions (a) and (b)*. For  $S$  satisfying conditions (a) and (b), let us now introduce the following two linear subspaces of  $X$ :

$$G_-(S; h) = \left\{ x \in X : \begin{array}{l} (\lambda - S)^{-1}x \text{ has an analytic extension for } \operatorname{Re} \lambda < h \\ \text{which vanishes in the norm as } \operatorname{Re} \lambda \rightarrow -\infty \end{array} \right\};$$

$$G_+(S; h) = \left\{ x \in X : \begin{array}{l} (\lambda - S)^{-1}x \text{ has an analytic extension for } \operatorname{Re} \lambda > -h \\ \text{which vanishes in the norm as } \operatorname{Re} \lambda \rightarrow +\infty \end{array} \right\}.$$

Then, by Liouville's theorem from complex analysis,  $G_+(S; h)$  and  $G_-(S; h)$  have zero intersection and do not depend on  $h$ , provided  $S$  satisfies conditions (a) and (b) above for this particular  $h > 0$ . In general,  $G_+(S; h)$  and  $G_-(S; h)$  need not be closed in  $X$ . So we let  $F_{\pm}(S; h)$  stand for the closure of  $G_{\pm}(S; h)$ . It is easily seen that an exponentially dichotomous operator  $S(X \rightarrow X)$  satisfies

$$F_+(S; h) = G_+(S; h) = \operatorname{Ker} P, \quad F_-(S; h) = G_-(S; h) = \operatorname{Im} P,$$

where  $P$  is the separating projection of the bisemigroup generated by  $S$ .

The following crucial proposition can be found in [16].

**Proposition 1.6.** *Let  $S(X \rightarrow X)$  be a closed and densely defined linear operator on the complex Banach space  $X$  satisfying the above conditions (a) and (b). Then*

the closed linear subspaces  $F_{\pm}(S; h)$  have zero intersection and add up to a dense linear subspace of  $X$ . Moreover,

$$F_+(S; h) \dot{+} F_-(S; h) = X \quad (1.8)$$

if and only if there exists a bounded linear operator  $P$  on  $X$  such that

$$Px = -\frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} (\lambda - S)^{-1} S^2 x \frac{d\lambda}{\lambda^2}, \quad x \in \mathcal{D}(S^2). \quad (1.9)$$

In that case  $P$  is the projection of  $X$  onto  $F_-(S; h)$  along  $F_+(S; h)$ .

If  $S$  is exponentially dichotomous, then the operator  $P$  given by (1.9) extends to a bounded linear operator on  $X$  and is indeed the separating projection of the bisemigroup generated by  $S$ . To see this, let us temporarily denote this separating projection by  $\tilde{P}$ . Then for  $x \in \mathcal{D}(S^2)$  we substitute (1.6) into (1.9) and compute

$$\begin{aligned} Px &= -\frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \int_{-\infty}^{\infty} e^{-\lambda t} E(t) S^2 x dt \frac{d\lambda}{\lambda^2} \\ &= \int_{-\infty}^{\infty} \left( \frac{-1}{2\pi i} \int_{h-i\infty}^{h+i\infty} e^{-\lambda t} \frac{d\lambda}{\lambda^2} \right) E(t) S^2 x dt, \end{aligned}$$

where the application of Fubini's theorem is justified by the exponential bounds on  $E(t)S^2x$ . For  $t > 0$  we replace the integral along  $h + i\mathbb{R}$  by a contour consisting of a semicircle in the right half-plane  $\operatorname{Re} \lambda \geq h$  and a segment of  $h + i\mathbb{R}$ , yielding zero for the integral. For  $t < 0$  we replace the integral along  $h + i\mathbb{R}$  by a contour consisting of a segment of  $h + i\mathbb{R}$  and a semicircle in the left half-plane  $\operatorname{Re} \lambda \leq h$ . As a result, we have for  $x \in \mathcal{D}(S^2)$ ,

$$\begin{aligned} Px &= - \int_{-\infty}^0 \left( \operatorname{Res}_{\lambda=0} \frac{e^{-\lambda t}}{\lambda^2} \right) E(t) S^2 x dt \\ &= \int_{-\infty}^0 t E(t) S^2 x dt = - \int_{-\infty}^0 t e^{tS_-} (S_-)^2 \tilde{P} x dt \\ &= - \left[ t e^{tS_-} S_- \tilde{P} x - e^{tS_-} \tilde{P} x \right]_{t=-\infty}^0 = \tilde{P} x, \end{aligned}$$

which proves that  $P$  extends to a bounded linear operator on  $X$  and is indeed the separating projection of the bisemigroup generated by  $S$ .

*Proof of Proposition 1.6. Part I.* Define the auxiliary linear operators

$$A_- = \frac{-1}{2\pi i} \int_{h-i\infty}^{h+i\infty} (\lambda - S)^{-1} \frac{d\lambda}{\lambda^2}, \quad A_+ = \frac{1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} (\lambda - S)^{-1} \frac{d\lambda}{\lambda^2}.$$

Then conditions (a) and (b) imply that  $A_-$  and  $A_+$  are well defined, belong to  $\mathcal{L}(X)$ , and commute with  $S$ . Further,

$$-A_+A_- = \left(\frac{1}{2\pi i}\right)^2 \int_{-h-i\infty}^{-h+i\infty} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda^2 \mu^2} (\lambda - S)^{-1} (\mu - S)^{-1} d\mu d\lambda$$

(using the resolvent identity)

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{-h-i\infty}^{-h+i\infty} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda^2 \mu^2 (\mu - \lambda)} [(\lambda - S)^{-1} - (\mu - S)^{-1}] d\mu d\lambda$$

(using Fubini's theorem twice)

$$\begin{aligned} &= \left(\frac{1}{2\pi i}\right)^2 \int_{-h-i\infty}^{-h+i\infty} \left( \int_{h-i\infty}^{h+i\infty} \frac{d\mu}{\mu^2 (\mu - \lambda)} \right) \frac{1}{\lambda^2} (\lambda - S)^{-1} d\lambda \\ &\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{h-i\infty}^{h+i\infty} \left( \int_{-h-i\infty}^{-h+i\infty} \frac{d\lambda}{\lambda^2 (\mu - \lambda)} \right) \frac{1}{\mu^2} (\mu - S)^{-1} d\mu, \end{aligned}$$

which vanishes identically. Here the application of Fubini's theorem is justified by the estimate

$$\left\| \frac{(\mu - S)^{-1}}{\lambda^2 \mu^2 (\mu - \lambda)} \right\| \leq \frac{\text{const.}}{2h|\lambda|^2|\mu|^2}.$$

Thus  $A_+A_- = 0$ . Similarly, we prove that  $A_-A_+ = 0$ . Using Cauchy's theorem and conditions (a) and (b) we easily verify that

$$\begin{aligned} A_+ + A_- &= \frac{-1}{2\pi i} \int_{h-i\infty}^{h+i\infty} (\lambda - S)^{-1} \frac{d\lambda}{\lambda^2} + \frac{1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} (\lambda - S)^{-1} \frac{d\lambda}{\lambda^2} \\ &= \frac{-1}{2\pi i} \oint_{|\lambda|=\frac{h}{2}} \frac{1}{\lambda^2} (\lambda - S)^{-1} d\lambda = S^{-2}, \end{aligned}$$

which is a bounded linear operator with zero kernel. As a result,

$$[\text{Ker } A_+ \cap \text{Ker } A_-] \subset \text{Ker } (S^{-2}) = \{0\}. \quad (1.10)$$

The bounded inverse  $T = S^{-1}$  of  $S$  commutes with  $A_+$  and  $A_-$ , and hence  $T[\text{Im } A_{\pm}] \subset \text{Im } A_{\pm}$ . Putting

$$M_{\pm} = \overline{\text{Im } A_{\pm}}, \quad (1.11)$$

we have  $T[M_{\pm}] \subset \mathcal{D}(S) \cap M_{\pm}$ , so that  $S$  maps  $T[M_{\pm}]$  into  $M_{\pm}$ . Then the restriction  $S_{\pm}$  of  $S$  to  $T[M_{\pm}]$  (acting from  $T[M_{\pm}]$  into  $M_{\pm}$ ) coincides with the (unbounded) inverse of the restriction of  $T$  to  $M_{\pm}$ , and therefore  $S_{\pm}(M_{\pm} \rightarrow M_{\pm})$  is closed.

Since  $M_{\pm} \subset \overline{T[M_{\pm}]}$  and  $\mathcal{D}(S)$  is dense in  $X$ , we conclude that  $\mathcal{D}(S_{\pm}) = T[M_{\pm}]$  is dense in  $M_{\pm}$ .

Let us now prove the inclusions

$$\sigma(S_+) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < -h\}, \quad (1.12a)$$

$$\sigma(S_-) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > h\}. \quad (1.12b)$$

Indeed, for  $z \in C_h = \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\}$  we have  $(z - S)^{-1}[M_{\pm}] \subset M_{\pm}$ . Because of  $(z - S)^{-1} = -T + zT(z - S)^{-1}$ , we also have  $(z - S)^{-1}[M_{\pm}] \subset T[M_{\pm}] = \mathcal{D}(S_{\pm})$ . Consequently,  $C_h \subset \rho(S_{\pm})$  and  $(z - S_{\pm})^{-1}$  coincides with the restriction of  $(z - S)^{-1}$  to  $M_{\pm}$ .

Now assume  $\operatorname{Re} z > h$  and put

$$R(z) = \frac{-1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{z^2}{\lambda^2(\lambda - z)} (\lambda - S)^{-1} d\lambda.$$

Then  $R(z)$  is bounded on  $X$  (as a result of conditions (a) and (b)), while the identity

$$(z - S) \left( \frac{z^2}{\lambda^2(\lambda - z)} (\lambda - S)^{-1} \right) = -\frac{z^2}{\lambda^2} (\lambda - S)^{-1} + \frac{z^2}{\lambda^2(\lambda - z)} I_X$$

implies that

$$-\frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} (z - S) \left( \frac{z^2}{\lambda^2(\lambda - z)} (\lambda - S)^{-1} \right) d\lambda = -z^2 A_- + I_X. \quad (1.13)$$

Since  $z - S$  is a closed operator, we can apply Lemma 1.2 to put  $z - S$  in front of the integral sign in (1.13) (which yields  $(z - S)R(z)$ ) and prove that  $(z - S)R(z) = -z^2 A_- + I_X$ . Hence for each  $w \in X$  we have

$$(z - S)R(z)A_+w = [-z^2 A_- + I_X] A_+w = A_+w. \quad (1.14)$$

In particular,  $(z - S_+)^{-1}$  coincides with the restriction of  $R(z)$  to  $M_+$ , which settles (1.12a). Assuming  $\operatorname{Re} z < -h$  and putting

$$\tilde{R}(z) = \frac{1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} \frac{z^2}{\lambda^2(\lambda - z)} (\lambda - S)^{-1} d\lambda, \quad (1.15)$$

we prove in the same way that  $(z - S)\tilde{R}(z) = -z^2 A_+ + I_X$  and hence that

$$(z - S)\tilde{R}(z)A_-w = [-z^2 A_+ + I_X] A_-w = A_-w, \quad w \in X.$$

Consequently, (1.12b) is true.

Now observe that, for any  $\delta \in (0, h)$ ,

$$R(z) = \frac{-1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{z^2}{\lambda^2(\lambda - z)} (\lambda - S)^{-1} d\lambda.$$



For  $x \in \mathcal{D}(S_+^2)$  we have

$$(z - S_+)^{-1}x = z^{-1}x + z^{-2}S_+x + z^{-2}(z - S_+)^{-1}S_+^2x,$$

where

$$\frac{1}{z^2}(z - S_+)^{-1}S_+^2x = \frac{1}{z^2}R(z)S_+^2x = \frac{-1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{1}{\lambda^2(\lambda - z)}(\lambda - S)^{-1}S_+^2x d\lambda,$$

which is bounded for  $\operatorname{Re} z > h$  (as a result of condition (b)). Thus for  $x \in \mathcal{D}(S_+^2)$  we have  $(z - S_+)^{-1}x$  bounded for  $\operatorname{Re} z > h$ . Further, by the Dominated Convergence Theorem, it vanishes as  $\operatorname{Re} z \rightarrow +\infty$ . Consequently,  $\mathcal{D}(S_+^2) \subset G_+(S; h)$ . In a similar way we prove that  $\mathcal{D}(S_-^2) \subset G_-(S; h)$ .

From  $A_-A_+ = A_+A_- = 0$  it is clear that  $M_- \subset \operatorname{Ker} A_+$  and  $M_+ \subset \operatorname{Ker} A_-$ . Since  $\operatorname{Ker} A_+ \cap \operatorname{Ker} A_- = \{0\}$ , we obtain  $M_+ \cap M_- = \{0\}$ .

To prove that  $M_+ + M_-$  is dense in  $X$ , we replace  $S$  by its dual operator  $S^*$  and observe that  $S^*$  is closed and densely defined and satisfies conditions (a) and (b) for the same  $h > 0$ . It is readily verified that

$$\begin{aligned} A_-^* &= \frac{-1}{2\pi i} \int_{h-i\infty}^{h+i\infty} (\lambda - S^*)^{-1} \frac{d\lambda}{\lambda^2}, \\ A_+^* &= \frac{1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} (\lambda - S^*)^{-1} \frac{d\lambda}{\lambda^2}. \end{aligned}$$

Then  $\operatorname{Ker} A_-^* \cap \operatorname{Ker} A_+^* = \{0\}$  and therefore  $\operatorname{Im} A_- + \operatorname{Im} A_+$  is dense in  $X$ . With the help of (1.11) we then get the density of  $M_+ + M_-$  in  $X$ .

From (1.12b) we know that  $S_+^{-1} \in \mathcal{L}(M_+)$  and  $S_+^{-1}$  has a dense range in  $M_+$ . But then  $S_+^{-2}$  has these same two properties, which means that  $\mathcal{D}(S_+^2)$  is dense in  $M_+$ . Since  $\mathcal{D}(S_\pm^2) \subset G_\pm(S; h)$  and  $\mathcal{D}(S_\pm^2)$  is dense in  $M_\pm$ , we obtain  $M_\pm \subset F_\pm(S; h)$ . As a result,

$$\overline{F_+(S; h) + F_-(S; h)} = X, \quad (1.16)$$

which concludes the proof of the first part.

*Part II.* For  $x \in \mathcal{D}(S^2)$  we define  $Px$  by (1.9). Then by comparing (1.9) with the definitions of  $A_\pm$  we get

$$A_-w = PS^{-2}w, \quad A_+w = (I - P)S^{-2}w, \quad (1.17)$$

where  $w \in X$ . If  $x \in \mathcal{D}(S_+^2)$ , then  $w = S^2x \in M_+ \subset \operatorname{Ker} A_-$  and hence  $Px = A_-w = 0$ . Moreover, for  $x \in \mathcal{D}(S_-^2)$  we have  $w = S^2x \in M_- \subset \operatorname{Ker} A_+$  and hence  $(I - P)x = A_+w = 0$ , implying that  $Px = x$ .

The boundedness of  $P$  (in the norm of  $X$ ) would imply that  $Px = 0$  for every  $x \in M_+$  (because  $Px = 0$  for  $x \in \mathcal{D}(S_+^2)$ ) and  $Px = x$  for every  $x \in M_-$  (because

$Px = x$  for  $x \in \mathcal{D}(S_-^2)$ ). Also, because  $M_+ \cap M_- = \{0\}$  and  $M_+ + M_-$  is dense in  $X$ , while  $P(I - P)x = 0$  for every  $x \in M_+ + M_-$ , we get  $P(I - P)x = 0$  for every  $x \in X$ . Consequently,  $P$  is a projection on  $X$  whose range contains the closed subspace  $M_-$  and whose kernel contains the closed subspace  $M_+$ . As a result,  $\text{Im } P = M_-$  and  $\text{Ker } P = M_+$ , and hence  $M_+ \dot{+} M_- = X$ .

Conversely, suppose  $M_+ \dot{+} M_- = X$ . Then the bounded projection of  $X$  onto  $M_-$  along  $M_+$  extends the linear operator defined on  $\mathcal{D}(S^2)$  by (1.9). Consequently, we have proved that the boundedness of  $P$  is equivalent with the decomposition  $M_+ \dot{+} M_- = X$ .

*Part III.* Recalling the definitions of  $G_+(S; h)$  and  $G_-(S; h)$ , it is clear from Cauchy's theorem that  $G_+(S; h) \subset \text{Ker } A_-$  and  $G_-(S; h) \subset \text{Ker } A_+$ . Therefore,

$$\mathcal{D}(S_+^2) \subset G_+(S; h) \subset \text{Ker } A_-, \quad \mathcal{D}(S_-^2) \subset G_-(S; h) \subset \text{Ker } A_+. \quad (1.18)$$

Taking closures we have

$$M_+ \subset F_+(S; h) \subset \text{Ker } A_-, \quad M_- \subset F_-(S; h) \subset \text{Ker } A_+,$$

which implies (1.16), as a result of the density of  $M_+ + M_-$  in  $X$ . Using (1.10) we now get  $F_+(S; h) \cap F_-(S; h) = \{0\}$ .

The boundedness of  $P$  implies that  $M_+ \dot{+} M_- = X$ , which in turn implies  $M_\pm = F_\pm(S; h)$  and the decomposition (1.8). Conversely, (1.8) implies that  $M_+ \dot{+} M_-$  is closed in  $X$  (while it is dense in  $X$ ) and therefore coincides with  $X$ . We have thus completed the proof of the second part of the statement of Proposition 1.6.  $\square$

We now derive the following characterization of exponentially dichotomous operators, in fact a minor variant of a result given in [16].

**Theorem 1.7.** *Let  $S$  be a closed and densely defined linear operator on the complex Banach space  $X$  satisfying conditions (a) and (b). Then  $S$  is exponentially dichotomous if and only if there exist  $E : \mathbb{R} \times X \rightarrow X$  and a constant  $r > 0$  such that for every  $x \in X$  we have  $e^{r|\cdot|}E(\cdot, x) \in L^\infty(\mathbb{R}; X)$  and*

$$\left\| e^{r|\cdot|}E(\cdot, x) \right\|_{L^\infty(\mathbb{R}, X)} \leq \text{const.} \|x\|,$$

and for some  $h > 0$  we have the Laplace transform formula

$$(\lambda - S)^{-1}x = \int_{-\infty}^{\infty} e^{-\lambda t} E(t, x) dt, \quad |\text{Re } \lambda| \leq h, \quad (1.19)$$

where  $x \in X$ .

*Proof.* In this proof we shall continue to use the notation introduced in the proof of Proposition 1.6. If  $S$  is exponentially dichotomous and  $E$  is the bisemigroup

generated by  $S$ , then  $E(t, x) \stackrel{\text{def}}{=} E(t)x$  obviously satisfies the conditions of Theorem 1.7.

Conversely, let the conditions of Theorem 1.7 be satisfied. Put

$$\Psi_+(\lambda, x) = \int_0^\infty e^{-\lambda t} E(t, x) dt, \quad \operatorname{Re} \lambda > -r, \quad (1.20a)$$

$$\Psi_-(\lambda, x) = \int_{-\infty}^0 e^{-\lambda t} E(t, x) dt, \quad \operatorname{Re} \lambda < r. \quad (1.20b)$$

Then for every  $x \in X$  we have

$$\|\Psi_+(\lambda, x)\| \leq \frac{\text{const.} \|x\|}{\operatorname{Re} \lambda + r}, \quad \operatorname{Re} \lambda > -r, \quad (1.21a)$$

$$\|\Psi_-(\lambda, x)\| \leq \frac{\text{const.} \|x\|}{r - \operatorname{Re} \lambda}, \quad \operatorname{Re} \lambda < r, \quad (1.21b)$$

while

$$(\lambda - S)^{-1}x = \Psi_+(\lambda, x) + \Psi_-(\lambda, x), \quad |\operatorname{Re} \lambda| < r, \quad x \in X.$$

As a result,  $S$  satisfies conditions (a) and (b) (for any  $h \in (0, r)$ ).

For  $x = S^{-2}w \in \mathcal{D}(S^2)$  we have

$$\begin{aligned} (\lambda - S)^{-1}x &= (\lambda - S_+)^{-1}(I_X - P)x + (\lambda - S_-)^{-1}Px \\ &= (\lambda - S_+)^{-1}A_+w + (\lambda - S_-)^{-1}A_-w, \end{aligned}$$

as a result of (1.14), (1.15), (1.17), and (1.18). Here the first term of the last member has an analytic continuation for  $\operatorname{Re} \lambda > -h$  and the second term of the last member has an analytic continuation for  $\operatorname{Re} \lambda < h$  [cf. (1.18)]. By Liouville's theorem, for  $x \in \mathcal{D}(S^2)$  we have

$$\Psi_+(\lambda, x) = \int_0^\infty e^{-\lambda t} E(t, x) dt = (\lambda - S_+)^{-1}(I_X - P)x, \quad \operatorname{Re} \lambda > -r, \quad (1.22a)$$

$$\Psi_-(\lambda, x) = \int_{-\infty}^0 e^{-\lambda t} E(t, x) dt = (\lambda - S_-)^{-1}Px, \quad \operatorname{Re} \lambda < r. \quad (1.22b)$$

We now extend  $P$  and  $I - P$  to all of  $X$  as follows:

$$(I - P)x = (\lambda - S)\Psi_+(\lambda, x), \quad Px = (\lambda - S)\Psi_-(\lambda, x),$$

where  $x \in X$  and  $|\operatorname{Re} \lambda| < r$ . Then  $(I - P)x$  and  $Px$  do not depend on  $\lambda$ . Using (1.22) and the fact that  $\lambda - S$  is a closed linear operator, we easily prove that  $P$  and  $I - P$  (with all of  $X$  as their domains) are closed linear operators. Indeed, if  $\|x_n - x\| \rightarrow 0$  and  $\|(\lambda - S)\Psi_+(\lambda, x_n) - y\| \rightarrow 0$ , then (i)  $\|\Psi_+(\lambda, x_n) - \Psi_+(\lambda, x)\| = \|\Psi_+(\lambda, x_n - x)\| \rightarrow 0$  (by (1.21a)), (ii)  $\Psi_+(\lambda, x_n) \in \mathcal{D}(S)$  (by (1.22a)), and hence  $\Psi_+(\lambda, x) \in \mathcal{D}(S)$  and  $(\lambda - S)\Psi_+(\lambda, x) = y$  (by  $\lambda - S$  being a closed operator). Thus,

by the Closed Graph Theorem,  $P$  is a bounded projection on  $X$ . As a result of the second part of Proposition 1.6, we obtain  $\text{Im } P = F_-(S; h)$  and  $\text{Ker } P = F_+(S; h)$ .

Repeated differentiation of (1.20) yields

$$\begin{aligned} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \lambda^n} \Psi_+(\lambda, x) &= \int_0^\infty \frac{t^n}{n!} e^{-\lambda t} E(t, x) dt, & \text{Re } \lambda > -r, \\ \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \lambda^n} \Psi_-(\lambda, x) &= \int_{-\infty}^0 \frac{t^n}{n!} e^{-\lambda t} E(t, x) dt, & \text{Re } \lambda < r, \end{aligned}$$

which coincides with  $(\lambda - S_+)^{-(n+1)}(I - P)x$  and  $(\lambda - S_-)^{-(n+1)}Px$ , respectively, when  $x \in \mathcal{D}(S^2)$ . For  $x \in \mathcal{D}(S^2)$  we thus find the estimates

$$\|(\lambda - S_+)^{-(n+1)}(I - P)x\| \leq \frac{\text{const.}\|x\|}{(\text{Re } \lambda + r)^{n+1}}, \quad \text{Re } \lambda > -r, \quad (1.23a)$$

$$\|(\lambda - S_-)^{-(n+1)}Px\| \leq \frac{\text{const.}\|x\|}{(r - \text{Re } \lambda)^{n+1}}, \quad \text{Re } \lambda < r. \quad (1.23b)$$

The estimates (1.23) imply that  $S_+$  and  $-S_-$  are infinitesimal generators of strongly continuous semigroups on  $F_+(S; h)$  and  $F_-(S; h)$ , respectively, with negative exponential growth bounds [cf. Theorem 1.3]. Considering these semigroups as the constituent semigroups of a strongly continuous bisemigroup  $E$  we obtain

$$(\lambda - S)^{-1}x = \int_{-\infty}^\infty e^{-\lambda t} E(t)x dt, \quad |\text{Re } \lambda| < r,$$

which means that  $E(t, x) = E(t)x$  for every  $x \in X$  and a.e.  $t \in \mathbb{R}$ . In other words,  $S$  is exponentially dichotomous.  $\square$

In order to express a bisemigroup in terms of the resolvent of its generator, we need to introduce the Cesaro mean

$$\begin{aligned} (C, 1) \int_{h-i\infty}^{h+i\infty} (\lambda - S)^{-1}x d\lambda &= \lim_{N \rightarrow \infty} \int_{-N}^N \left(1 - \frac{|\nu|}{N}\right) (h + i\nu - S)^{-1}x i d\nu \\ &= \lim_{N \rightarrow \infty} \int_0^N \frac{dl}{N} \int_{-l}^l (h + i\nu - S)^{-1}x i d\nu. \end{aligned} \quad (1.24)$$

Now let  $X$  be a complex Banach space and let  $\Phi$  be a locally (Bochner) integrable function on  $\mathbb{R}^+$  with values in  $X$ . Denote by

$$f(\lambda) = \int_0^\infty e^{-\lambda t} \Phi(t) dt$$

the Laplace transform of  $\Phi$ . If we denote by  $\sigma_a(\Phi)$  the abscissa of absolute convergence of the Laplace transform of  $\Phi$ ,

$$\sigma_a(\Phi) = \inf \left\{ \sigma \in \mathbb{R} : \int_0^\infty \|e^{-\lambda t} \Phi(t)\| dt < \infty \text{ for } \text{Re } \lambda > \sigma \right\},$$

then for  $c > \sigma_a(\Phi)$  we have

$$\frac{1}{2\pi i}(C, 1) \int_{c-i\infty}^{c+i\infty} e^{\lambda t} f(\lambda) d\lambda = \begin{cases} 0, & t < 0, \\ \frac{1}{2}\Phi(0^+), & t = 0, \\ \frac{1}{2}[\Phi(t^+) + \Phi(t^-)], & t > 0, \end{cases} \quad (1.25)$$

whenever the expressions in the right-hand side of (1.25) have meaning. This is the case, for instance, if  $\Phi$  is locally of bounded variation. Equation (1.25) follows from Theorem II.9.2 of [164] after applying continuous linear functionals to either side.

Applying (1.25) we obtain

$$\frac{1}{2\pi i}(C, 1) \int_{h-i\infty}^{h+i\infty} e^{\lambda t} (\lambda - S)^{-1} (I - P)x d\lambda = \begin{cases} 0, & t < 0, \\ \frac{1}{2}(I - P)x, & t = 0, \\ E(t)x, & t > 0, \end{cases} \quad (1.26a)$$

$$\frac{1}{2\pi i}(C, 1) \int_{h-i\infty}^{h+i\infty} e^{\lambda t} (\lambda - S)^{-1} Px d\lambda = \begin{cases} 0, & t > 0, \\ -\frac{1}{2}Px, & t = 0, \\ E(t)x, & t < 0, \end{cases} \quad (1.26b)$$

for any  $h \in (-r, r)$  with  $r > 0$  as in the statement of Theorem 1.3, which implies

$$\frac{1}{2\pi i}(C, 1) \int_{h-i\infty}^{h+i\infty} e^{\lambda t} (\lambda - S)^{-1} x d\lambda = \begin{cases} E(t)x, & t \neq 0, \\ \frac{1}{2}(I - 2P)x, & t = 0, \end{cases} \quad (1.27)$$

and hence

$$Px = \frac{1}{2}x - \frac{1}{2\pi i}(C, 1) \int_{h-i\infty}^{h+i\infty} (\lambda - S)^{-1} x d\lambda, \quad x \in X. \quad (1.28)$$

The representations (1.26)–(1.28) have been employed in [98] to derive a characterization of infinitesimal generators of hyperbolic semigroups on a general Banach space.

## 1.4 Classes of exponential dichotomy

In this section we introduce special classes of exponentially dichotomous operators and bisemigroups by imposing constraints on the constituent semigroups. For these special classes we derive characterizations as well as necessary conditions and sufficient conditions.

### 1.4.1 Analytic bisemigroups

Analytic semigroups have been discussed in detail in [86, 103, 127, 118, 8, 60]. In [86, 103, 8] the term “holomorphic semigroup” is used. In [118] analytic semigroups are assumed bounded by default. Here we introduce bisemigroups composed of analytic semigroups of negative exponential growth bound.

By a *bounded analytic semigroup* (of angle  $\delta \in (0, \frac{\pi}{2})$ ) on a complex Banach space  $X$  we mean a bounded strongly continuous function

$$E : \overline{\Sigma_\delta} = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| \leq \zeta\} \cup \{0\} \rightarrow \mathcal{L}(X)$$

having the following properties:

- 1) The restriction of  $E$  to  $\Sigma_\delta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \zeta\}$  is analytic.
- 2) We have the semigroup property

$$E(t+s) = E(t)E(s), \quad t, s \in \overline{\Sigma_\delta}; \quad E(0) = I_X.$$

A closed linear operator  $S$  defined on a complex Banach space  $X$  is called *sectorial* if there exists  $\delta \in (0, \frac{\pi}{2}]$  such that the sector

$$\Sigma_{\frac{\pi}{2}+\delta} \stackrel{\text{def}}{=} \left\{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \frac{\pi}{2} + \delta \right\}$$

is contained in the resolvent set of  $S$  and if for each  $\varepsilon \in (0, \delta)$  there exists  $M_\varepsilon \geq 1$  such that

$$\|(\lambda - S)^{-1}\| \leq \frac{M_\varepsilon}{|\lambda|}, \quad 0 \neq \lambda \in \overline{\Sigma_{\frac{\pi}{2}+\delta-\varepsilon}}.$$

According to Theorem II 4.6 of [60], the sectorial operators are exactly the infinitesimal generators of bounded analytic semigroups.

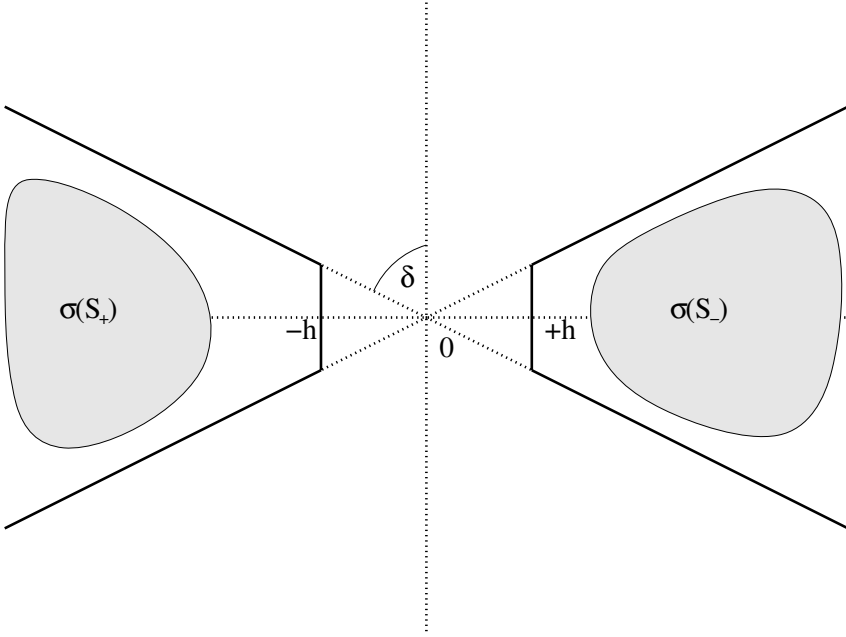
In order to generalize bounded analytic semigroups and their infinitesimal generators to the realm of bisemigroups, we define analytic bisemigroups and bisectorial operators as follows. By an *analytic bisemigroup* on a complex Banach space  $X$  we mean a strongly continuous bisemigroup on  $X$  whose constituent semigroups are bounded analytic. By a *bisectorial operator* we mean a closed and densely defined linear operator  $S$  on a complex Banach space  $X$  such that for certain  $\delta \in (0, \frac{\pi}{2}]$  and  $h > 0$  the set

$$\left\{ 0 \neq \lambda \in \mathbb{C} : \frac{\pi}{2} - \delta < |\arg(\lambda)| < \frac{\pi}{2} + \delta \right\} \cup \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| < h\} \quad (1.29)$$

is contained in the resolvent set of  $S$  [See Figure 1.1] and if for each  $\varepsilon \in (0, \delta)$  there exists  $M_\varepsilon \geq 1$  such that

$$\|(\lambda - S)^{-1}\| \leq \frac{M_\varepsilon}{|\lambda|}, \quad \frac{\pi}{2} - \varepsilon \leq |\arg(\lambda)| \leq \frac{\pi}{2} + \varepsilon. \quad (1.30)$$

The proof of the converse part of the following proposition has been inspired by the proof of [60, Proposition I 4.3].

Figure 1.1: Spectrum of a bisectorial operator  $S$ .

**Proposition 1.8.** *The infinitesimal generator of an analytic bisemigroup is bisectorial and any bisectorial operator generates an analytic bisemigroup.*

*Proof.* If  $S$  is the infinitesimal generator of an analytic bisemigroup  $E$  on  $X$  and  $P$  is the separating projection of  $E$ , then the restrictions of  $E(\cdot)$  to  $\text{Ker } P$  and of  $E(\cdot)$  to  $\text{Im } P$  are bounded analytic semigroups on  $\text{Ker } P$  and  $\text{Im } P$ . Thus their generators  $S_+$  and  $-S_-$  are sectorial operators: There exists  $\delta \in (0, \frac{\pi}{2}]$  such that  $\Sigma_{\frac{\pi}{2}+\delta} \subset \rho(S_+) \cap \rho(-S_-)$  and for each  $\varepsilon \in (0, \delta)$  there exists  $M_\varepsilon \geq 1$  such that

$$\max(\|(\lambda - S_+)^{-1}\|, \|(\lambda + S_-)^{-1}\|) \leq \frac{M_\varepsilon}{|\lambda|}, \quad 0 \neq \lambda \in \overline{\Sigma_{\frac{\pi}{2}+\delta-\varepsilon}}.$$

On the other hand, there exists  $h > 0$  such that

$$\{\lambda \in \mathbb{C} : |\text{Re } \lambda| < h\} \subset \rho(S). \quad (1.31)$$

Therefore,  $S$  satisfies (1.29) and (1.30) and hence is bisectorial.

Conversely, let  $S(X \rightarrow X)$  be bisectorial and let  $\delta \in (0, \frac{\pi}{2}]$  and  $M_\varepsilon > 0$  be the constants in (1.29) and (1.30). For  $\delta_1 \in (0, \delta)$  and  $\varepsilon \in (0, h)$  we define the curve  $\gamma(\delta_1, \varepsilon)^+$  consisting of the union of the half-lines  $\{\lambda \in \mathbb{C} : \arg(\lambda + \varepsilon) = (\pi/2) + \delta_1\}$  and  $\{\lambda \in \mathbb{C} : \arg(\lambda + \varepsilon) = (3\pi/2) - \delta_1\}$ , oriented by running from infinity to  $-\varepsilon$

passing through the third quadrant and then from  $-\varepsilon$  to infinity passing through the second quadrant. For sufficiently small  $\varepsilon > 0$  we consider

$$F(t) = \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^+} e^{\lambda t} (\lambda - S)^{-1} d\lambda, \quad (1.32)$$

where  $|\arg(t)| < |\delta_1|$ . Then for  $t = |t|e^{i\phi}$  with  $|\phi| < \min(\delta_1, \pi - \delta_1)$  we have

$$|e^{\lambda t}| = e^{-\varepsilon \operatorname{Re} t} e^{(\lambda + \varepsilon)t \cos(\arg((\lambda + \varepsilon)t))} = e^{-\varepsilon \operatorname{Re} t} e^{-|(\lambda + \varepsilon)t| \sin(\delta_1 \pm \phi)}, \quad (1.33)$$

where  $\arg(\lambda + \varepsilon) = \pi \mp ((\pi/2) - \delta_1)$ . Also note the existence of  $M > 0$  such that  $\|(\lambda - S)^{-1}\| \leq (M/|\lambda|)$  for each  $\lambda$  with  $|\frac{\pi}{2} - \arg(\lambda)| \leq \delta_1$ , as well as the fact that the integral in (1.32) does not depend on  $\varepsilon > 0$  provided it is small enough. Thus by taking  $\varepsilon = \operatorname{const} \cdot |t|$  for  $|t|$  small enough, we can prove that the integral in (1.32) is absolutely convergent in the operator norm, uniformly in  $t \in \mathbb{C}$  satisfying  $|\arg(t)| \leq \zeta |\delta_1|$  for any  $\zeta \in (0, 1)$ . Thus  $F(t)$  is analytic in  $t$  for  $|\arg(t)| < \delta_1$ . Moreover,

$$\|F(t)\| \leq \frac{M}{2\pi} \int_{\gamma(\delta_1, \varepsilon)^+} \frac{e^{-\varepsilon \operatorname{Re} t} e^{-|(\lambda + \varepsilon)t| \sin((1-\zeta)\delta_1)}}{|\lambda|} d|\lambda|, \quad |\arg(t)| \leq \zeta \delta_1,$$

whenever  $\zeta \in (0, 1)$ . Since  $0 \notin \gamma(\delta_1, \varepsilon)^+$ , the operator function  $F(t)$  is bounded in  $t$  with  $|\arg(t)| \leq \zeta \delta_1$  for each  $\zeta \in (0, 1)$ .

In the same way we introduce the curve  $\gamma(\delta_1, \varepsilon)^-$  consisting of the union of the half-lines  $\{\lambda \in \mathbb{C} : \arg(\lambda - \varepsilon) = (\pi/2) + \delta_1\}$  and  $\{\lambda \in \mathbb{C} : \arg(\lambda - \varepsilon) = (3\pi/2) - \delta_1\}$ , oriented by running from infinity to  $\varepsilon$  passing through the fourth quadrant and from  $-\varepsilon$  to infinity passing through the first quadrant and prove that, for sufficiently small  $\varepsilon > 0$ ,

$$F(t) = \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^-} e^{\lambda t} (\lambda - S)^{-1} d\lambda \quad (1.34)$$

is absolutely convergent in the operator norm, uniformly in  $t \in \mathbb{C}$  satisfying  $|\pi - \arg(t)| \leq \zeta |\delta_1|$  for any  $\zeta \in (0, 1)$ , which implies the analyticity of  $F(t)$  in  $t$  for  $|\pi - \arg(t)| < \delta_1$ . In the same way we prove the boundedness of  $F(t)$  in  $t$  with  $|\pi - \arg(t)| \leq \zeta \delta_1$  for each  $\zeta \in (0, 1)$ .

For  $x = S^{-1}w \in \mathcal{D}(S)$  we have

$$\begin{aligned} [F(t) - F(0^\pm)]x &= \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^\pm} \frac{e^{\lambda t} - 1}{\lambda} ((\lambda - S)^{-1}w + S^{-1}w) d\lambda \\ &= \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^\pm} \frac{e^{\lambda t} - 1}{\lambda} (\lambda - S)^{-1}w d\lambda + \left( \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^\pm} \frac{e^{\lambda t} - 1}{\lambda} d\lambda \right) S^{-1}w \\ &= \left( \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^\pm} \frac{e^{\lambda t} - 1}{\lambda} d\lambda \right) S^{-1}w = 0, \end{aligned}$$



where the first integral vanishes as a result of the Theorem of Dominated Convergence (using that  $|\lambda^{-1}[e^{\lambda t} - 1]| \leq |t|e^{-\varepsilon|t|}$ ) and the second integral by closing the contour by a circular arc and applying Cauchy's theorem. The uniform boundedness of  $F(t)$  as  $t \rightarrow 0^\pm$  then implies the strong convergence of  $F(t)$  to  $F(0^\pm)$  as  $t \rightarrow 0^\pm$ .

Next, letting  $t, \tau$  have their arguments in  $(-\delta_1, \delta_1)$ , we use the resolvent equation and Fubini's theorem in the following calculation:

$$\begin{aligned}
F(t)F(\tau) &= \frac{1}{(2\pi i)^2} \int_{\gamma(\delta_1, \varepsilon)^+} \int_{\gamma(\delta_1, (\varepsilon/2)^+)} e^{\lambda t} e^{\mu \tau} (\mu - S)^{-1} (\lambda - S)^{-1} d\mu d\lambda \\
&= \frac{1}{(2\pi i)^2} \int_{\gamma(\delta_1, \varepsilon)^+} \int_{\gamma(\delta_1, (\varepsilon/2)^+)} \frac{e^{\lambda t} e^{\mu \tau}}{\lambda - \mu} [(\mu - S)^{-1} - (\lambda - S)^{-1}] d\mu d\lambda \\
&= \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon/2)^+} e^{\mu \tau} \left( \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^+} \frac{e^{\lambda t}}{\lambda - \mu} d\lambda \right) (\mu - S)^{-1} d\mu \\
&\quad - \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^+} e^{\lambda t} \left( \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon/2)^+} \frac{e^{\mu \tau}}{\lambda - \mu} d\mu \right) (\lambda - S)^{-1} d\lambda \\
&= -\frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^+} e^{\lambda t} (-e^{\lambda \tau}) (\lambda - S)^{-1} d\lambda \\
&= F(t + \tau)x.
\end{aligned}$$

On the other hand, letting  $t, \tau$  have their arguments in  $(\pi - \delta_1, \pi + \delta_1)$ , we compute in an analogous way

$$\begin{aligned}
F(t)F(\tau) &= \frac{1}{(2\pi i)^2} \int_{\gamma(\delta_1, \varepsilon)^-} \int_{\gamma(\delta_1, (\varepsilon/2)^-)} e^{\lambda t} e^{\mu \tau} (\mu - S)^{-1} (\lambda - S)^{-1} d\mu d\lambda \\
&= \frac{1}{(2\pi i)^2} \int_{\gamma(\delta_1, \varepsilon)^-} \int_{\gamma(\delta_1, (\varepsilon/2)^-)} \frac{e^{\lambda t} e^{\mu \tau}}{\lambda - \mu} [(\mu - S)^{-1} - (\lambda - S)^{-1}] d\mu d\lambda \\
&= \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon/2)^-} e^{\mu \tau} \left( \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^-} \frac{e^{\lambda t}}{\lambda - \mu} d\lambda \right) (\mu - S)^{-1} d\mu \\
&\quad - \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^-} e^{\lambda t} \left( \frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon/2)^-} \frac{e^{\mu \tau}}{\lambda - \mu} d\mu \right) (\lambda - S)^{-1} d\lambda \\
&= -\frac{1}{2\pi i} \int_{\gamma(\delta_1, \varepsilon)^-} e^{\lambda t} (e^{\lambda \tau}) (\lambda - S)^{-1} d\lambda \\
&= -F(t + \tau)x.
\end{aligned}$$

We have thus derived the bisemigroup property, while the exponential decay of  $\|F(t)\|$  is clear from (1.33) and the strong continuity as  $t \rightarrow 0^\pm$  has been established above. Hence  $F : \mathbb{R} \rightarrow \mathcal{L}(X)$  is a strongly continuous (and in fact analytic) bisemigroup. It remains to determine its infinitesimal generator.

Indeed, using that  $\|(\lambda - S)^{-1}\| \leq (M/|\lambda|)$  for some  $M > 0$  whenever either  $|\frac{\pi}{2} - \arg(\lambda)| \leq \delta_1$  or  $|\frac{3\pi}{2} - \arg(\lambda)| \leq \delta_1$ , we get for the integral along the segments  $\{s \pm i\beta : 0 \leq s \leq \beta \tan \delta_1\}$ ,

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int e^{\lambda t} (\lambda - S)^{-1} d\lambda \right\| &\leq \frac{M}{2\pi} \int_0^{\beta \tan \delta_1} \frac{e^{s|t| \cos \delta_1}}{\sqrt{\beta^2 + s^2}} ds \\ &= \frac{M}{2\pi} \int_0^{\tan \delta_1} \frac{e^{-\beta \sigma \cos \delta_1}}{\sqrt{1 + \sigma^2}} d\sigma, \end{aligned}$$

which vanishes as  $\beta \rightarrow +\infty$ . As a result of Cauchy's theorem we get

$$F(t)x = \begin{cases} \frac{1}{2\pi i} \int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} e^{\lambda t} (\lambda - S)^{-1} x d\lambda, & |\arg(t)| < \delta_1, \\ \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{\lambda t} (\lambda - S)^{-1} x d\lambda, & |\pi - \arg(t)| < \delta_1. \end{cases}$$

Equations (1.26) then show that  $S$  is indeed the infinitesimal generator of  $F$ , as claimed. Consequently,  $S$  generates an analytic bisemigroup on  $X$ .  $\square$

### 1.4.2 Immediately norm continuous bisemigroups

A strongly continuous semigroup  $E : [0, \infty) \rightarrow \mathcal{L}(X)$  on a complex Banach space  $X$  is called *immediately norm continuous* if its restriction to  $(0, \infty)$  is norm continuous. Such semigroups have the property that

$$\lim_{\tau \rightarrow \pm\infty} \|(\sigma + i\tau - S)^{-1}\| = 0, \quad \sigma > \omega(E), \quad (1.35)$$

where  $S$  stands for the infinitesimal generator of  $E$  (cf. Corollary II 4.19 of [60]). In fact, writing

$$(\lambda - S)^{-1} = \int_0^\infty e^{-\lambda t} E(t) dt, \quad \operatorname{Re} \lambda > \omega(E),$$

where the integral is well defined in the operator norm, it is a simple matter to apply the Riemann-Lebesgue Lemma to the function  $e^{-\sigma(\cdot)} E(\cdot) \in L^1(\mathbb{R}^+, \mathcal{L}(X))$  to prove (1.35).

A strongly continuous bisemigroup is called *immediately norm continuous* if its constituent semigroups are immediately norm continuous. Its infinitesimal generator  $S$  has the property that there exists  $h > 0$  such that  $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\} \subset \rho(S)$  and  $\|(\lambda - S)^{-1}\| \rightarrow 0$  as  $|\lambda| \rightarrow \infty$  within this strip.

We have the following sufficient condition.

**Theorem 1.9.** *Let  $X$  be a complex Hilbert space. Let  $S(X \rightarrow X)$  be an exponentially dichotomous operator such that for some  $h > 0$  the set*

$$C_h = \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\} \subset \rho(S)$$

and  $\|\lambda(\lambda - S)^{-1}\|$  is bounded on  $C_h$ . Then  $S$  generates an immediately norm continuous bisemigroup on  $X$ .

*Proof.* When converting the estimate of  $(\lambda - S)^{-1}$  into estimates of the resolvents of the infinitesimal generators of the constituent semigroups, we can apply Theorem II 4.20 of [60] and prove these semigroups to be immediately norm continuous. But then the bisemigroup itself is immediately norm continuous.  $\square$

In Chapter 8 we shall introduce a class of exponentially dichotomous operators  $A$  on the Banach space  $X = C([-q, p]; \mathbb{C}^M)$  such that  $\|\lambda(\lambda - A)^{-1}\|$  is bounded on  $C_h$  for some  $h > 0$ . Nevertheless, the corresponding constituent semigroups are translation semigroups on a suitable domain of continuous vector functions and therefore fail to be immediately norm continuous. In other words, Theorem 1.9 does not extend to a general Banach space.

### 1.4.3 Immediately compact bisemigroups

A strongly continuous semigroup  $E : [0, \infty) \rightarrow \mathcal{L}(X)$  on a complex Banach space  $X$  is called *immediately compact* if  $E(t)$  is a compact operator for each  $t > 0$ . Then it is known [60, Theorem 4.29] that  $E$  is immediately compact iff  $E$  is immediately norm continuous and its infinitesimal generator has a compact resolvent.

We now call a strongly continuous bisemigroup *immediately compact* if its constituent semigroups are immediately compact. We then have the following obvious result.

**Theorem 1.10.** *A strongly continuous bisemigroup on a complex Banach space is immediately compact iff it is immediately norm continuous and its infinitesimal generator has a compact resolvent on the imaginary axis.*

In Chapter 8 we shall deal with a class of exponentially dichotomous operators with compact resolvent which are neither immediately compact nor immediately norm continuous.

### 1.4.4 Hyperbolic semigroups

A strongly continuous semigroup  $T : [0, \infty) \rightarrow X$  is called *hyperbolic* if there exists a bounded projection  $P$  on  $X$  such that

$$T(t)[\operatorname{Im} P] \subset \operatorname{Im} P, \quad T(t)[\operatorname{Ker} P] \subset \operatorname{Ker} P, \quad (1.36)$$

the semigroup  $T_+$  defined by restricting  $T$  to  $\operatorname{Ker} P$  satisfies  $\omega(T_+) < 0$ , the semigroup  $T_-$  defined by restricting  $T$  to  $\operatorname{Im} P$  extends to a strongly continuous group on  $\operatorname{Im} P$ , and the semigroup defined by inverting  $T_-(-\cdot)$  satisfies  $\omega(T_-(-\cdot)^{-1}) < 0$ .

Let  $S(X \rightarrow X)$  be the infinitesimal generator of the semigroup  $S$ . Applying the Laplace transform formula to the semigroups  $T_+$  and  $T_-(-\cdot)^{-1}$ , it appears

that there exists a strip  $C_h = \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\}$  for some  $h > 0$  which is contained in  $\rho(S)$  and on which  $(\lambda - S)^{-1}$  is bounded. Defining  $E(t)$  by

$$E(t) = \begin{cases} T_+(t)(I - P)x, & t > 0, \\ -(T_-(-t)|_{\operatorname{Ker} P})^{-1}, & t < 0, \end{cases} \quad (1.37)$$

we see that  $E$  is a strongly continuous bisemigroup on  $X$  with infinitesimal generator  $S$ . Moreover, for  $t < 0$  the restrictions of  $E(t)$  to  $\operatorname{Im} P$  are invertible operators on  $\operatorname{Im} P$ .

Conversely, let  $E$  be a strongly continuous bisemigroup on  $X$  with separating projection  $P$  such that for  $t < 0$  the restrictions of  $E(t)$  to  $\operatorname{Im} P$  are invertible operators on  $\operatorname{Im} P$ . Then

$$T(t) = E(t) - (E(-t)|_{\operatorname{Im} P})^{-1}P, \quad t > 0, \quad (1.38)$$

is a hyperbolic semigroup on  $X$ . Thus infinitesimal generators of hyperbolic semigroups can be viewed as a special kind of exponentially dichotomous operator. Consequently, (1.9) is the exact expression for the projection  $P$  in the definition of a hyperbolic semigroup.

We summarize the above (and attach a minor result to it) as follows [98]:

**Proposition 1.11.** *Suppose  $S(X \rightarrow X)$  is the infinitesimal generator of a strongly continuous semigroup  $T$  on  $X$ . Then the following statements are equivalent:*

- a.  $S$  is exponentially dichotomous.
- b.  $T$  is a hyperbolic semigroup.
- c. There exists  $\delta \in (0, 1)$  such that

$$\{\lambda \in \mathbb{C} : 1 - \delta < |\lambda| < 1 + \delta\} \subset \rho(T(1)). \quad (1.39)$$

Moreover, in that case the separating projection of the bisemigroup generated by  $S$  coincides with the Riesz projection of  $T(1)$  corresponding to the part of its spectrum outside the unit disk.

*Proof.* The first two parts have been shown to be equivalent above. Assuming that  $T$  is hyperbolic, there exist constants  $M, \alpha, \beta > 0$  such that

$$\begin{aligned} \|T_+(t)\| &\leq M e^{-\alpha t}, \\ \|T_-(t)^{-1}\| = \|T_-(-t)\| &\leq M e^{-\beta t}, \end{aligned}$$

where  $t > 0$ . Hence,

$$\|T_+(1)^n\|^{1/n} \leq M^{1/n} e^{-\alpha} < 1, \quad \|T_-(1)^{-n}\|^{1/n} \leq M^{1/n} e^{-\beta} < 1,$$

for sufficiently large  $n \in \mathbb{N}$ . Consequently,

$$\sigma(T(1)) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq e^{-\alpha}\} \cup \{\lambda \in \mathbb{C} : |\lambda| \geq e^{\beta}\}.$$

Conversely, let (1.39) be true. Let

$$I - P = \frac{1}{2\pi i} \oint_{\lambda \in \mathbb{T}} (\lambda - T(1))^{-1} d\lambda$$

be the Riesz projection of  $T(1)$  corresponding to its spectrum within the unit disk. Let  $T_+(t)$  and  $T_-(t)$  be the restrictions of  $T(t)$  to  $\text{Ker } P$  and  $\text{Im } P$ , respectively. Then  $T_-(t)$  is necessarily invertible on  $\text{Im } P$  and  $T_-(-t) = T_-(t)^{-1}$ . Since the spectral radii  $\rho_+$  and  $\rho_-$  of  $T_+(1)$  and  $T_-(-1) = T_-(1)^{-1}$  are strictly less than 1, we get for  $p, q \in \mathbb{N}$ ,

$$\begin{aligned} \|T_+(\frac{p}{q})^n\|^{1/n} &\leq \|T_+(np)\|^{1/nq} \rightarrow \rho_+^{p/q}, \\ \|T_-(-\frac{p}{q})^n\|^{1/n} &\leq \|T_-(-np)\|^{1/nq} \rightarrow \rho_-^{p/q}, \end{aligned}$$

which implies that  $\omega(T_+) \leq \log(\rho_+) < 0$  and  $\omega(T_-(-\cdot)) \leq \log(\rho_-) < 0$ . Hence,  $T$  is hyperbolic.  $\square$

On complex Hilbert spaces, it is known [66, 85] that an infinitesimal generator  $S$  of a semigroup whose resolvent  $(\lambda - S)^{-1}$  exists and is bounded on a strip  $\{\lambda \in \mathbb{C} : |\text{Re } \lambda| \leq h\}$  for some  $h > 0$ , is exponentially dichotomous.

The following example shows that this characterization of a hyperbolic semigroup generator does not hold in a general Banach space [7, 79].

**Example 1.12.** Let  $X = C_0[0, \infty) \cap L^1([0, \infty), e^s ds)$ , where  $C_0[0, \infty)$  is the complex Banach space of continuous complex-valued functions on  $[0, \infty)$  vanishing at infinity and the norm on  $X$  is given by

$$\|f\|_X = \sup_{s \in [0, \infty)} |f(s)| + \int_0^\infty e^s |f(s)| ds.$$

Then

$$(T(t)f)(\tau) = f(t + \tau)$$

defines a strongly continuous semigroup on  $X$  satisfying

$$[(\lambda - S)^{-1}f](\tau) = \int_0^\infty e^{-\lambda t} [T(t)f](\tau) dt = \int_0^\infty e^{-\lambda t} f(t + \tau) dt.$$

Then for  $\text{Re } \lambda > -1$  and  $f \in X$  we estimate

$$\begin{aligned} \|(\lambda - S)^{-1}f\|_X &\leq \sup_{\tau \geq 0} \int_0^\infty e^{-t \text{Re } \lambda} |f(t + \tau)| dt \\ &\quad + \int_0^\infty e^\tau \int_0^\infty e^{-t \text{Re } \lambda} |f(t + \tau)| dt d\tau \leq \left(1 + \frac{1}{1 + \text{Re } \lambda}\right) \|f\|_X. \end{aligned}$$

Furthermore, applying  $T(t)$  to a positive function  $f \in X$  satisfying  $f(0) = f(t) = \|f\|_\infty = 1$  and  $\int_0^\infty e^s f(s) ds < \varepsilon$ , we prove that  $\|f\|_X < 1 + \varepsilon$  and  $\|T(t)f\|_X > 1$ . Therefore,  $\|T(t)\| = 1$  for any  $t \geq 0$ . Consequently,  $T$  is not a hyperbolic semigroup on  $X$ , in spite of the existence and boundedness of  $(\lambda - S)^{-1}$  for  $|\text{Re } \lambda| \leq \frac{1}{2}$ .

## 1.5 Adjoint and sun dual bisemigroups

If  $E : [0, \infty) \rightarrow \mathcal{L}(X)$  is a strongly continuous semigroup on a complex Banach space  $X$ , then the Banach dual  $E(\cdot)^*$  has the semigroup property, but need not be strongly continuous. On the other hand, the Banach dual of a strongly continuous semigroup on a *reflexive* Banach space and in particular on a complex Hilbert space is a strongly continuous semigroup. For semigroups on general Banach spaces, there exists a “maximal” restriction of  $E(\cdot)^*$ , the so-called sun dual  $E^\odot(\cdot)$  of  $E(\cdot)$ , which is strongly continuous. Here we define sun dual semigroups, discussed in detail in [160, 60], and generalize them to the bisemigroup setting.

Let  $E : [0, \infty) \rightarrow \mathcal{L}(X)$  be a strongly continuous semigroup on a complex Banach space  $X$ , and let  $X^*$  be the dual space of  $X$ . Let us define the operator  $A^\odot(X^* \rightarrow X^*)$  by

$$\left\{ \begin{array}{l} \mathcal{D}(A^\odot) = \left\{ x^* \in X^* : \begin{array}{l} \exists y^* \in X^* \text{ such that} \\ \lim_{t \rightarrow 0^+} \left\| \frac{E(t)^* x^* - x^*}{t} - y^* \right\|_{X^*} = 0 \end{array} \right\}, \\ A^\odot x^* = y^*. \end{array} \right. \quad (1.40)$$

Now let  $X^\odot$  be the closure of  $\mathcal{D}(A^\odot)$  in  $X^*$ . Then

$$E(t)^*[X^\odot] \subset X^\odot, \quad t \in \mathbb{R}^+,$$

while the restriction

$$E^\odot(t) = E(t)^*|_{X^\odot}$$

of  $E(t)^*$  to  $X^\odot$  defines a strongly continuous semigroup, the so-called *sun dual semigroup*, whose infinitesimal generator coincides with the restriction of  $A^\odot$  to  $X^\odot$ .

Now let  $E : \mathbb{R} \rightarrow \mathcal{L}(X)$  be a strongly continuous bisemigroup on a complex Banach space  $X$ , and let  $P = -E(0^-)$  be its separating projection. Let us define the domain  $\mathcal{D}(A^\odot)$  of the sun dual bisemigroup generator by

$$\mathcal{D}(A^\odot) = \left\{ x^* \in X^* : \begin{array}{l} \exists y_\pm^* \in X^* \text{ such that} \\ \lim_{t \rightarrow 0^\pm} \left\| \frac{E(t)^* x^* - x^*}{t} - y_\pm^* \right\|_{X^*} = 0 \end{array} \right\}. \quad (1.41)$$

Let  $X^\odot$  be the closure of  $\mathcal{D}(A^\odot)$  in  $X^*$ . Then  $X^\odot$  is a closed linear subspace of  $X^*$  and  $A^\odot(X^\odot \rightarrow X^\odot)$  is defined by

$$A^\odot x^* = y_+^* + y_-^*. \quad (1.42)$$

Moreover,

$$E(t)^*[X^\odot] \subset X^\odot, \quad t \in \mathbb{R} \text{ or } t = 0^\pm,$$

while the restriction

$$E^\odot(t) = E(t)^*|_{X^\odot}$$

of  $E(t)^*$  to  $X^\odot$  defines a strongly continuous bisemigroup, the so-called *sun dual bisemigroup*, whose infinitesimal generator coincides with  $A^\odot$ . If  $X$  is a complex Hilbert space or, more generally, a complex reflexive Banach space, then  $X^\odot$  coincides with the dual space  $X^*$  and hence the sun dual bisemigroup with the dual bisemigroup.

## Chapter 2

# Perturbing Exponentially Dichotomous Operators

In this chapter our ultimate goal is to prove (or disprove) that bounded additive perturbations  $S$  of exponentially dichotomous operators  $S_0$  are exponentially dichotomous, provided there exists a vertical strip of the form

$$\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq \varepsilon\}$$

within the resolvent set of the perturbed operator  $S$  and the resolvent  $(\lambda - S)^{-1}$  is bounded on this strip. Although at first sight the solution of this perturbation problem seems to be a piece of cake (as it obviously is in the semigroup case), the dependence of the separating projection of  $S$  on the perturbation considerably complicates the problem. Our basic strategy is to derive the perturbed bisemigroup by solving a vector-valued convolution equation on the line using the unperturbed bisemigroup as an integral kernel and the unperturbed bisemigroup acting on an arbitrary vector as the inhomogeneous term. This requires representing pointwise inverses of Fourier transforms of operator-valued functions as Fourier transforms of operator-valued functions. We therefore first discuss basic Gelfand theory of commutative Banach algebras and derive the classical Allan-Bochner-Phillips theorem on inversion within the operator-valued Wiener algebra. We then develop additive perturbation theory if the perturbation is a compact operator or the bisemigroup is analytic (or at least immediately norm continuous). Here we can remain within the comfortable realm of Bochner integrals of vector-valued functions. To deal with arbitrary bounded perturbations, we cast the Allan-Bochner-Phillips theorem in tensor product language before generalizing it to deal with bounded additive perturbations of exponentially dichotomous operators that either have a sufficiently small norm or act on complex Hilbert spaces. For convenience we review both Bochner and Pettis integration of vector-valued functions. The most



general bounded additive perturbation result on arbitrary complex Banach spaces still eludes us.

## 2.1 Invertibility in Banach algebras

In this section we review invertibility of elements of commutative as well as non-commutative Banach algebras. For more detailed information we refer to a variety of textbooks [67, 70, 58].

### 2.1.1 Commutative Banach algebras

Invertibility theory in complex commutative Banach algebras  $Z$  with unit element  $e$  is well understood. Let us denote by  $M$  the set of all multiplicative linear functionals on  $Z$ , i.e., the continuous linear functionals  $\varphi : Z \rightarrow \mathbb{C}$  which are not trivial (i.e.,  $\varphi \not\equiv 0$ ) and obey the product rule

$$\varphi(zw) = \varphi(z)\varphi(w), \quad z, w \in Z.$$

For any  $\varphi \in M$ ,  $\{x \in Z : \varphi(x) = 0\}$  is a maximal ideal in  $Z$ . Conversely, for every maximal ideal  $m$  in  $Z$ , the functional  $\varphi : Z \rightarrow \mathbb{C}$  specifying, for each  $z \in Z$ , the unique complex number  $\varphi(z)$  such that  $z - \varphi(z)e$  belongs to  $m$ , is multiplicative. An element  $z \in Z$  is invertible in  $Z$  if and only if  $\varphi(z) \neq 0$  for every  $\varphi \in M$ . The set  $M$  can be naturally imbedded as a closed subset into the closed unit ball in the dual space of  $Z$  equipped with the weak-\* topology. According to Alaoglu's theorem [58, Theorem 1.23], the closed unit ball with weak-\* topology is a compact Hausdorff space. Hence  $M$  is a compact Hausdorff space called the *maximal ideal space* of  $Z$ .

As a first example, consider the commutative Banach algebra  $\ell^1(\mathbb{Z})$  of all complex sequences  $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$  for which

$$\|\mathbf{x}\| = \sum_{n=-\infty}^{\infty} |x_n|$$

is finite. This algebra is known as the (discrete) *Wiener algebra*. Then the multiplicative linear functionals are the discrete Fourier transform maps

$$\mathbf{x} \mapsto \hat{\mathbf{x}}(\zeta) = \sum_{n=-\infty}^{\infty} \zeta^n x_n,$$

where  $\zeta$  belongs to the unit circle  $\mathbb{T}$ . An element  $\mathbf{x} \in \ell^1(\mathbb{Z})$  is invertible if and only if  $\hat{\mathbf{x}}(\zeta) \neq 0$  for every  $\zeta \in \mathbb{T}$ .

An example of the utmost importance to this book is the (*continuous*) Wiener algebra. This is the complex Banach algebra  $\mathbb{C} \dot{+} L^1(\mathbb{R})$  with norm

$$\|(c, f)\| \stackrel{\text{def}}{=} |c| + \int_{-\infty}^{\infty} |f(t)| dt$$

and convolution product

$$(c_1, f_1) * (c_2, f_2) = (c_1 c_2, c_2 f_1 + c_1 f_2 + (f_1 * f_2)),$$

where  $(f_1 * f_2)(t) = \int_{-\infty}^{\infty} f_1(t-s)f_2(s) ds$ . Then the multiplicative linear functionals are the Fourier transform maps

$$\widehat{(c, f)}(\lambda) = c + \int_{-\infty}^{\infty} e^{i\lambda t} f(t) dt,$$

where  $\lambda \in \mathbb{R}$ , plus the functional  $(c, f) \mapsto c$  which can be thought of as the value of the Fourier transform of  $(c, f)$  at  $\pm\infty$ . An element  $(c, f)$  of the Wiener algebra is invertible if and only if  $c \neq 0$  and the Fourier transform  $\widehat{(c, f)}(\lambda) \neq 0$  for each  $\lambda \in \mathbb{R}$ .

## 2.1.2 Noncommutative Banach algebras

Let  $\mathcal{A}$  be a complex Banach algebra with unit element  $e$ ; in general,  $\mathcal{A}$  is not commutative. Let  $Z$  be a commutative Banach subalgebra of  $\mathcal{A}$  such that  $e \in Z$  and  $az = za$  for  $a \in \mathcal{A}$  and  $z \in Z$ . Let  $\mathcal{F}$  be a closed subalgebra of  $\mathcal{A}$ . Then by  $Z \otimes \mathcal{F}$  we denote the algebraic tensor product of  $Z$  and  $\mathcal{F}$ , namely the set of all finite sums

$$\sum_{j=1}^n z_j f_j,$$

where  $n \in \mathbb{N}$ ,  $\{z_1, \dots, z_n\} \subset Z$ , and  $\{f_1, \dots, f_n\} \subset \mathcal{F}$ . We assume that  $Z \otimes \mathcal{F}$  is a dense linear subspace of  $\mathcal{A}$ .

Let  $M$  be the maximal ideal space of  $Z$ . For every  $\varphi \in M$ , we define  $\Phi_\varphi : Z \otimes \mathcal{F} \rightarrow \mathcal{A}$  as follows:

$$\Phi_\varphi \left( \sum_{j=1}^n z_j f_j \right) = \sum_{j=1}^n \varphi(z_j) f_j. \quad (2.1)$$

Then  $\mathcal{A}$  is said to be *realizable as a tensor product* of  $Z$  and  $\mathcal{F}$  if and only if  $\Phi_\varphi$  extends to a bounded linear operator on  $\mathcal{A}$  for each  $\varphi \in M$ . It is clear that  $\Phi_\varphi[\mathcal{A}] \subset \mathcal{F}$ ,  $(\Phi_\varphi)^2 = \Phi_\varphi$ , and  $\Phi_\varphi(xy) = \Phi_\varphi(x)\Phi_\varphi(y)$  for  $x, y \in \mathcal{A}$ . We therefore call  $\Phi_\varphi$  the *multiplicative projection* associated with the multiplicative linear functional  $\varphi$ .

The following theorem is due to Bochner and Phillips [33] in the case where  $\mathcal{A}$  is a Banach function algebra and to Allan [6] in general. We follow the presentation given in the Appendix of [75], which is based on [6].

**Theorem 2.1 (Allan-Bochner-Phillips).** *Let  $\mathcal{A}$  be a complex Banach algebra with unit element  $e$  realized as a tensor product of a commutative subalgebra  $Z$  and some subalgebra  $\mathcal{F}$ , where  $e \in Z$ . Let  $M$  denote the maximal ideal space of  $Z$ . Then  $A \in \mathcal{A}$  is invertible in  $\mathcal{A}$  if and only if  $\Phi_\varphi(A)$  is invertible in  $\mathcal{F}$  for each  $\varphi \in M$ .*

In Theorem 2.1 we may replace “invertible” by either “left invertible” or “right invertible.”

Before proving Theorem 2.1, we discuss the principal application we have in mind. Let  $\mathcal{B}$  be some Banach algebra with unit element and let

$$\mathcal{A} \stackrel{\text{def}}{=} \mathcal{B} \dot{+} L^1(\mathbb{R}; \mathcal{B})$$

be endowed with the norm

$$\|(B, F)\| \stackrel{\text{def}}{=} \|B\| + \int_{-\infty}^{\infty} \|F(t)\| dt,$$

where  $B \in \mathcal{B}$  and  $F \in L^1(\mathbb{R}; \mathcal{B})$ . The product operation in  $\mathcal{A}$  is defined as

$$(B_1, F_1) * (B_2, F_2) = (B_1 B_2, B_1 F_2(\cdot) + F_1(\cdot) B_2 + F_1 * F_2),$$

where  $(F_1 * F_2)(t) = \int_{-\infty}^{\infty} F_1(t-s) F_2(s) ds$ . For  $Z$  we take the commutative subalgebra

$$Z = \{(cI_{\mathcal{B}}, f(\cdot)I_{\mathcal{B}}) : c \in \mathbb{C}, f \in L^1(\mathbb{R})\},$$

which obviously contains the identity  $(I_{\mathcal{B}}, 0)$  of  $\mathcal{A}$ . Then each element of  $Z$  commutes with each element of  $\mathcal{A}$ . For  $\mathcal{F}$  we take the closed subalgebra

$$\mathcal{F} = \{(B, 0) : B \in \mathcal{B}\}.$$

Then  $Z \otimes \mathcal{F}$  can be identified with the set of elements  $(B, F)$ , where  $B \in \mathcal{B}$  and  $F(t) = \sum_{j=1}^n (c_j + f_j(t)) B_j$  for a.e.  $t \in \mathbb{R}$ ,  $n \in \mathbb{N}$ , and certain  $c_1, \dots, c_n \in \mathbb{C}$ ,  $f_1, \dots, f_n \in L^1(\mathbb{R})$ , and  $B_1, \dots, B_n \in \mathcal{B}$ . Since we can take characteristic functions of a set of pairwise disjoint subsets of  $\mathbb{R}$  of finite measure as our  $f_j$ , the general theory of Bochner integration (cf. Subsection 2.3.2) yields that  $Z \otimes \mathcal{F}$  is dense in  $\mathcal{A}$ . Since the Fourier transform maps  $(c, f) \xrightarrow{\mathcal{F}} c + \hat{f}(\lambda)$  yield the multiplicative linear functionals of  $Z$ , we get

$$\Phi_\varphi \left( (B, \sum_{j=1}^n (c_j, f_j) B_j) \right) = B + \sum_{j=1}^n c_j B_j + \sum_{j=1}^n \hat{f}_j(\lambda) B_j,$$

which obviously extends to the bounded linear operator

$$(B, F) \mapsto B + \int_{-\infty}^{\infty} e^{i\lambda t} F(t) dt$$

on  $\mathcal{A}$ . Also, the multiplicative linear functional  $(c, f) \xrightarrow{\psi} c$  induces the multiplicative projection  $\Phi_\psi$  defined by

$$\Phi_\psi \left( (B, \sum_{j=1}^n (c_j, f_j) B_j) \right) = B + \sum_{j=1}^n c_j B_j,$$

which obviously extends to the bounded linear operator  $(B, F) \mapsto B$  on  $\mathcal{A}$ . Thus  $(B, F)$  is invertible in  $\mathcal{A}$  if and only if  $B$  and all of the Fourier transforms

$$B + \int_{-\infty}^{\infty} e^{i\lambda t} F(t) dt, \quad \lambda \in \mathbb{R},$$

are invertible in  $\mathcal{B}$ .

A second application regards the complex Banach algebra  $\ell^1(\mathbb{Z}; \mathcal{B})$  of all sequences  $\{B_n\}_{n=-\infty}^{\infty}$  with entries  $B_n$  in some complex Banach algebra  $\mathcal{B}$  with unit element  $e_{\mathcal{B}}$  endowed with the norm

$$\|\{B_n\}_{n=-\infty}^{\infty}\| \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} \|B_n\|_{\mathcal{B}}$$

and the convolution product

$$\{B_n^{(1)}\}_{n=-\infty}^{\infty} * \{B_n^{(2)}\}_{n=-\infty}^{\infty} \stackrel{\text{def}}{=} \left\{ \sum_{m=-\infty}^{\infty} B_{n-m}^{(1)} B_m^{(2)} \right\}_{n=-\infty}^{\infty}.$$

Applying Theorem 2.1 for  $\mathcal{A} = \ell^1(\mathcal{B})$ ,  $Z = \ell^1(\mathbb{Z}) \otimes e_{\mathcal{B}}$ , and  $\mathcal{F}$  the subalgebra of sequences  $\{B_n\}_{n=-\infty}^{\infty}$  satisfying  $B_n = 0$  for  $0 \neq n \in \mathbb{Z}$ , we easily see that  $Z \otimes \mathcal{F}$  contains the  $\mathcal{B}$ -valued sequences with at most finitely many nonzero terms and hence is dense in  $\mathcal{A}$ . Consequently, a sequence  $\{B_n\}_{n=-\infty}^{\infty}$  is invertible in  $\ell^1(\mathbb{Z}; \mathcal{B})$  if and only if its so-called symbol

$$\hat{B}(z) = \sum_{n=-\infty}^{\infty} z^n B_n,$$

is invertible in  $\mathcal{B}$  for every  $z \in \mathbb{T}$ .

Before deriving Theorem 2.1, we prove the following lemma [6, 75].

**Lemma 2.2 (Allan).** *Let  $\mathcal{A}$  be a complex Banach algebra with unit element  $e$  and let  $Z$  be a closed commutative subalgebra of  $\mathcal{A}$  such that  $e \in Z$  and  $az = za$  for  $a \in \mathcal{A}$  and  $z \in Z$ . If  $L$  is a maximal left ideal in  $\mathcal{A}$  or a maximal right ideal in  $\mathcal{A}$ , then  $L \cap Z$  is a maximal ideal in  $Z$ .*

*Proof.* Let  $L$  be a maximal left ideal in  $\mathcal{A}$ . Then, clearly,  $L \cap Z$  is a two-sided ideal in  $Z$ . To show that it is in fact a maximal ideal in  $Z$ , we define for every  $z \in Z \setminus L$ ,

$$K_z = \{y \in \mathcal{A} : yz \in L\}.$$

Then  $K_z$  is a left ideal in  $\mathcal{A}$  with  $e \notin K_z$ . Because for  $z \in Z \setminus L$  and  $y \in L$  we have  $yz = zy \in L$ , we also have  $L \subset K_z$ . The maximality of  $L$  then implies that  $K_z = L$  for every  $z \in Z \setminus L$ . This implies that if for  $z \in Z \setminus L$  the elements  $y_1 z - e$  and  $y_2 z - e$  belong to  $L$ , then  $(y_1 - y_2)z \in L$  and therefore  $y_1 - y_2 \in K_z = L$ . On the other hand, from the maximality of the left ideal  $L$  in  $\mathcal{A}$  it follows that for each  $a \in Z \setminus L$  there exists  $y \in \mathcal{A}$  such that  $yz - e \in L$ . Consequently, for every  $z \in Z \setminus L$  there exists a unique element  $y \in \mathcal{A}$  modulo  $L$  such that  $yz - e \in L$ .

Now let  $z \in Z \setminus L$  be such that

$$\{z - \lambda e : \lambda \in \mathbb{C}\} \subset Z \setminus L.$$

Then for each  $\lambda \in \mathbb{C}$  there exists a unique  $y(\lambda) \in \mathcal{A}$  modulo  $L$  such that  $y(\lambda)(z - \lambda e) - e \in L$ . For fixed  $\lambda_0 \in \mathbb{C}$  the function

$$f(\lambda) \stackrel{\text{def}}{=} y(\lambda_0)(e - (\lambda - \lambda_0)y(\lambda_0))^{-1}$$

is analytic in a neighborhood  $U$  of  $\lambda_0$ . Since for all  $\lambda \in \mathbb{C}$  the element

$$g(\lambda) = z - \lambda e = (e - (\lambda - \lambda_0)y(\lambda_0))(z - \lambda_0 e) + (\lambda - \lambda_0)\ell \in L,$$

where  $\ell(\lambda) = y(\lambda_0)(z - \lambda_0 e) - e \in L$ , we have

$$\begin{aligned} g(\lambda)(z - \lambda e) - e &= y(\lambda_0)(e - (\lambda - \lambda_0)y(\lambda_0))^{-1}[(e - (\lambda - \lambda_0)y(\lambda_0))(z - \lambda_0 e) \\ &\quad + (\lambda - \lambda_0)\ell] = y(\lambda_0)(z - \lambda_0 e) + \ell_1, \end{aligned}$$

where  $\lambda \in U$  and  $\ell_1 = (\lambda - \lambda_0)y(\lambda_0)(e - (\lambda - \lambda_0)y(\lambda_0))^{-1}\ell \in L$ . Then the equivalence classes of  $\mathcal{A}/L$  containing the functions  $y(\lambda)$  and  $g(\lambda)$  coincide for  $\lambda \in U$ . Consequently, the equivalence class of  $\mathcal{A}/L$  containing  $y(\lambda)$  is an entire function of  $\lambda \in \mathbb{C}$  which has the zero equivalence class as its limit as  $|\lambda| \rightarrow \infty$ . Liouville's theorem then implies that  $y(\lambda) \in L$  for each  $\lambda \in \mathbb{C}$ . For  $\lambda = 0$  this yields  $zy(0) - e \in L$ , while always  $zy(0) = y(0)z \in L$ , implying that  $e \in L$ . Contradiction.

Consequently, for each  $z \in Z \setminus L$  there exists a necessarily unique  $\lambda \in \mathbb{C}$  such that  $z - \lambda e \in L$ . But this means that  $L \cap Z$  is a maximal ideal in  $Z$ .

If  $L$  is assumed to be a maximal right ideal in  $\mathcal{A}$ , the proof is similar.  $\square$

Let us now prove Theorem 2.1.

*Proof of Theorem 2.1.* If  $a \in \mathcal{A}$  is left invertible in  $\mathcal{A}$ , then  $ya = e$  for some  $y \in \mathcal{A}$ . Now let  $\Phi$  be a multiplicative projection. Then  $e = \Phi(e) = \Phi(ya) = \Phi(y)\Phi(a)$  and hence  $\Phi(a)$  is left invertible in  $\mathcal{F}$  with left inverse  $\Phi(y)$ .

Conversely, assume  $a \in \mathcal{A}$  is not left invertible. Then

$$K = \{xa : x \in \mathcal{A}\}$$

is a left ideal in  $\mathcal{A}$  such that  $e \notin K$ . Let  $L$  be a maximal left ideal in  $\mathcal{A}$  containing  $K$ . Then, according to Lemma 2.2, the set  $L \cap Z$  is a maximal ideal in  $Z$ . Let  $\varphi$  denote the multiplicative linear functional on  $Z$  such that

$$L \cap Z = \{z \in Z : \varphi(z) = 0\}.$$

Denote by  $\Phi$  the multiplicative projection associated with  $\varphi$ . For all elements  $x = \sum_{j=1}^n z_j f_j \in Z \otimes \mathcal{F}$  we have

$$x - \Phi(x) = \sum_{j=1}^n (z_j - \varphi(z_j)e) f_j = \sum_{j=1}^n f_j (z_j - \varphi(z_j)e) \in L.$$

Because  $\Phi$  is bounded on  $\mathcal{A}$ ,  $Z \otimes \mathcal{F}$  is dense in  $\mathcal{A}$ , and  $L$  is closed in  $\mathcal{A}$ , we obtain  $x - \Phi(x) \in L$  for each  $x \in \mathcal{A}$ . As a result,  $\text{Ker } \Phi \subset L$ .

Let us now prove that  $\Phi(xa) \neq e$  for each  $x \in \mathcal{A}$ . Otherwise there would exist  $x \in \mathcal{A}$  such that  $\Phi(xa - e) = \Phi(xa) - \Phi(e) = \Phi(xa) - e = 0$  and hence  $xa - e \in L$ . Since  $xa \in K$  and  $K \subset L$ , we get  $e \in L$ , which is a contradiction. Thus  $\Phi(xa) \neq e$  for every  $x \in \mathcal{A}$ .

Since  $\Phi(xa) \neq e$  for every  $x \in \mathcal{A}$ , there does not exist a left inverse of  $\Phi(a)$ , i.e., an element  $y \in \mathcal{A}$  such that  $y\Phi(a) = e$ . Otherwise we would have

$$e = \Phi(e) = \Phi(y\Phi(a)) = \Phi(y)\Phi^2(a) = \Phi(y)\Phi(a) = \Phi(ya),$$

which cannot be true. Thus  $\Phi(a)$  is not left invertible in  $\mathcal{A}$ .

For right invertibility (and hence two-sided invertibility) the proof is similar.  $\square$

## 2.2 Additive perturbations: Elementary results

Let  $S_0$  be an exponentially dichotomous operator on a complex Banach space  $X$ , and let  $\Gamma \in \mathcal{L}(X)$ . Put  $S = S_0 + \Gamma$ , which implies that  $\mathcal{D}(S) = \mathcal{D}(S_0)$ . Then for  $S$  to be exponentially dichotomous, there should exist  $h > 0$  such that

$$\{\lambda \in \mathbb{C} : |\text{Re } \lambda| \leq h\} \subset \rho(S). \quad (2.2)$$

Under the hypotheses of the present Section 2.2 the resolvent  $(\lambda - S)^{-1}$  should be bounded on the strip, potentially for a smaller  $h > 0$ . The question is if this condition is sufficient for the exponential dichotomy of  $S$ . In this section we impose additional constraints on  $\Gamma$  or on the bisemigroup generated by  $S_0$  to make  $S$  exponentially dichotomous. In Section 2.3 we shall prove that **any** bounded additive perturbation  $S$  of an exponentially dichotomous operator  $S_0$  on a complex Hilbert space is exponentially dichotomous if (2.2) is satisfied and  $(\lambda - S)^{-1}$  is bounded on the strip given by (2.2), but at the expense of dealing with Pettis (rather than Bochner) integrals of vector-valued functions. We have not managed to prove (or

disprove) the corresponding Banach space result. On the positive side, we shall prove that any additive perturbation of an exponentially dichotomous operator by a linear operator of sufficiently small norm is exponentially dichotomous.

### 2.2.1 Additive compact perturbations

Let us first deal with additive compact perturbations of exponentially dichotomous operators by proving a result in [134] which has been generalized in [157] to perturbations such that  $(\lambda - S_0)^{-1}\Gamma$  is a compact operator.

**Theorem 2.3.** *Let  $S_0$  be an exponentially dichotomous operator on  $X$  and let  $\Gamma$  be a compact operator on  $X$  without zero or purely imaginary eigenvalues. Then  $S = S_0 + \Gamma$  is exponentially dichotomous. Moreover, for  $0 \neq t \in \mathbb{R}$  and for  $t = 0^\pm$ , the operator  $E(t; S) - E(t; S_0)$  is compact.*

*Proof.* Consider

$$W(\lambda) = (\lambda - S_0)^{-1}(\lambda - S) = I_X - (\lambda - S_0)^{-1}\Gamma. \quad (2.3)$$

Then there exists  $h > 0$  such that  $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\} \subset \rho(S_0)$ . Because of the Laplace transform formula

$$(\lambda - S_0)^{-1}x = \int_{-\infty}^{\infty} e^{-\lambda t} E(t; S_0)x \, dt, \quad x \in X, \quad (2.4)$$

where  $E(\cdot; S_0)$  is the bisemigroup generated by  $S_0$ , we have  $\|(\lambda - S_0)^{-1}x\| \rightarrow 0$  as  $|\lambda| \rightarrow \infty$  with  $|\operatorname{Re} \lambda| < -\omega(E(\cdot; S_0))$ , for any  $x \in X$ . The compactness of  $\Gamma$  ensures that  $\|W(\lambda) - I_X\| = \|(\lambda - S_0)^{-1}\Gamma\| \rightarrow 0$  as  $|\lambda| \rightarrow \infty$  with  $|\operatorname{Re} \lambda| < -\omega(E(\cdot; S_0))$ . Thus for any  $r \in (0, -\omega(E(\cdot; S_0)))$  there exists  $s > 0$  such that  $W(\lambda)$  is invertible whenever  $|\operatorname{Re} \lambda| \leq r$  and  $|\operatorname{Im} \lambda| \geq s$ . Since the imaginary axis does not contain any eigenvalues of  $S$  and therefore  $W(\lambda)$  is invertible for  $\lambda$  on the imaginary axis, there exists  $\rho > 0$  such that  $W(\lambda)$  is invertible whenever  $|\operatorname{Re} \lambda| \leq \rho$ .

Now note that

$$W(\lambda)x = x - \int_{-\infty}^{\infty} e^{-\lambda t} E(t; S_0)\Gamma x \, dt, \quad x \in X.$$

Then the compactness of  $\Gamma$  guarantees that  $E(\cdot; S_0)\Gamma$  is continuous in the operator norm with a jump discontinuity in  $t = 0$ . The exponential decay of  $\|e^{ct}E(t; S_0)\|$  as  $t \rightarrow \pm\infty$  (for  $c \in [-\rho, \rho]$ ) implies that  $e^{c(\cdot)}E(\cdot; S_0)\Gamma \in L^1(\mathbb{R}; \mathcal{L}(X))$  for each  $c \in [-\rho, \rho]$ . Moreover,  $W(\lambda)$  is invertible for all  $\lambda \in -c + i\mathbb{R}$  (including at  $\lambda = -c \pm i\infty$ ). According to Theorem 2.1 there exists  $\Phi \in L^1(\mathbb{R}; \mathcal{L}(X))$  such that  $e^{c(\cdot)}\Phi \in L^1(\mathbb{R}; \mathcal{L}(X))$  for each  $c \in [-\rho, \rho]$  and

$$W(\lambda)^{-1} = I_X + \int_{-\infty}^{\infty} e^{-\lambda t} \Phi(t) \, dt, \quad |\operatorname{Re} \lambda| \leq \rho.$$

Consequently, for  $x \in X$  we have

$$\begin{aligned} (\lambda - S)^{-1}x &= W(\lambda)^{-1}(\lambda - S_0)^{-1}x \\ &= \int_{-\infty}^{\infty} e^{-\lambda t} \left[ E(t; S_0)x + \int_{-\infty}^{\infty} \Phi(t - \tau)E(\tau; S_0)x d\tau \right] dt, \end{aligned}$$

where  $|\operatorname{Re} \lambda| \leq \rho$ . Putting

$$E(t; S)x = E(t; S_0)x + \int_{-\infty}^{\infty} \Phi(t - \tau)E(\tau; S_0)x d\tau, \quad (2.5)$$

we obtain, for each  $x \in X$ , a continuous function with jump discontinuity in  $t = 0$  such that

$$\|E(t; S)x\| \leq \operatorname{const} e^{-\rho|t|}\|x\|, \quad x \in X.$$

Theorem 1.7 then implies that  $S$  is exponentially dichotomous.

To prove the compactness of  $E(t; S) - E(t; S_0)$ , we write (2.5) as

$$E(t; S) = E(t; S_0) + \int_{-\infty}^{\infty} \Phi(t - \tau)E(\tau; S_0) d\tau, \quad (2.6)$$

where the integral is absolutely convergent in the norm topology. Then

$$\int_{-\infty}^{\infty} e^{-\lambda t} \Phi(t) dt = \left[ I_X - \int_{-\infty}^{\infty} e^{-\lambda t} E(t; S_0) \Gamma dt \right]^{-1} - I_X = (\lambda - S)^{-1} \Gamma,$$

where  $|\operatorname{Re} \lambda| \leq \rho$ , which implies that  $\Phi(t) = E(t; S)\Gamma$  for  $0 \neq t \in \mathbb{R}$  and that  $\Phi(t)$  is a compact operator. But then (2.6) implies the compactness of  $E(t; S) - E(t; S_0)$ , as claimed.  $\square$

### 2.2.2 Additive perturbations: Analytic bisemigroups

We now prove that bounded additive perturbations of analytic bisemigroup generators are themselves analytic bisemigroup generators [134]. Our proof differs from the one given in [134].

**Theorem 2.4.** *Let  $S_0$  be the infinitesimal generator of an analytic bisemigroup and let  $\Gamma$  be bounded, both defined on the complex Banach space  $X$ . Suppose  $S = S_0 + \Gamma$  satisfies*

$$\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\} \subset \rho(S)$$

*for some  $h > 0$ . Then  $S$  is the infinitesimal generator of an analytic bisemigroup.*

*Proof.* Since  $S_0$  is bisectorial, there exist  $\delta \in (0, \frac{\pi}{2}]$ ,  $h > 0$ , and  $M_\varepsilon \geq 1$  such that (1.29) and (1.30) hold true. Now define the operator function  $W$  by (2.3). Then  $W(\lambda)$  is defined for  $\lambda \in \mathbb{C}$  as in (1.29) and  $\|W(\lambda) - I_X\| \rightarrow 0$  as  $|\lambda| \rightarrow \infty$  with  $\lambda$



as in (1.30). Then there exists  $\delta_1 \in (0, \delta] \subset (0, \frac{\pi}{2})$  such that  $W(\lambda)$  is invertible for  $\lambda$  belonging to the set

$$\left\{0 \neq \lambda \in \mathbb{C} : \frac{\pi}{2} - \delta_1 < |\arg(\lambda)| < \frac{\pi}{2} + \delta_1\right\} \cup \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| < h\}. \quad (2.7)$$

Thus the set in (2.7) is contained in  $\rho(S)$ . Moreover, if needed, we choose  $\delta_1$  and  $h$  as to make  $\|W(\lambda)^{-1}\|$  bounded on this set. Then the identity

$$(\lambda - S)^{-1} = W(\lambda)^{-1}(\lambda - S_0)^{-1}$$

implies that  $S$  is bisectorial. Proposition 1.8 then implies that  $S$  generates an analytic bisemigroup.  $\square$

### 2.2.3 Additive perturbations: Immediate norm continuity

We now prove that bounded additive perturbations of infinitesimal generators of immediately norm continuous bisemigroups are themselves infinitesimal generators of immediately norm continuous semigroups [134].

**Theorem 2.5.** *Let  $S_0$  be the infinitesimal generator of an immediately norm continuous bisemigroup and let  $\Gamma$  be bounded, both defined on the complex Banach space  $X$ . Suppose  $S = S_0 + \Gamma$  satisfies*

$$\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\} \subset \rho(S)$$

*for some  $h > 0$ . Then  $S$  is the infinitesimal generator of an immediately norm continuous bisemigroup.*

*Proof.* We essentially follow the proof of Theorem 2.3, using that  $E(\cdot; S_0)\Gamma \in L^1(\mathbb{R}, \mathcal{L}(X))$ . Then (2.5) yields the bisemigroup generated by  $S$ , where  $E(\cdot; S) - E(\cdot; S_0)$  is easily seen to be norm continuous (as the convolution product of  $\Phi \in L^1(\mathbb{R}, \mathcal{L}(X))$  and  $E(\cdot; S_0)$ ) except for a strong jump discontinuity in  $t = 0$ .  $\square$

### 2.2.4 Compact perturbations: Immediate compactness

We now prove that compact perturbations of infinitesimal generators of immediately compact bisemigroups are themselves infinitesimal generators of immediately compact bisemigroups.

**Theorem 2.6.** *Let  $S_0$  be the infinitesimal generator of an immediately compact bisemigroup and let  $\Gamma$  be a compact operator, both defined on the complex Banach space  $X$ . Suppose  $S = S_0 + \Gamma$  does not have imaginary spectrum. Then  $S$  is the infinitesimal generator of an immediately compact bisemigroup.*

*Proof.* According to Theorem 1.10,  $S_0$  is the infinitesimal generator of an immediately norm continuous bisemigroup. Then  $S$  generates an immediately norm

continuous bisemigroup (as a result of Theorem 2.5) and is a strongly continuous bisemigroup such that  $E(t; S) - E(t; S_0)$  is a compact operator for  $0 \neq t \in \mathbb{R}$  (as a result of Theorem 2.3). But then  $E(\cdot; S)$  is an immediately compact bisemigroup, as claimed.  $\square$

## 2.3 Additive perturbations: Generalizations

In this section we ultimately prove that a bounded additive perturbation of an exponentially dichotomous operator of sufficiently small norm or acting on an arbitrary complex Hilbert space is exponentially dichotomous, provided there exists a strip about the imaginary axis contained in the resolvent set of the perturbed operator and the resolvent of the perturbed operator is bounded on this strip. The method of proof is basically the same as in Section 2.2, but the vector-valued convolution integrals appearing in the proof can no longer be interpreted as Bochner integrals. We therefore introduce reasonable cross norms and the injective and projective tensor products of two Banach spaces [57] and cast the Allan-Bochner-Phillips theorem in the framework of tensor products. To deal with vector-valued integrals, we introduce the much weaker Pettis integral of vector-valued functions [57]. We then go on to interpret the completion of the vector space of Pettis integrable vector functions as an injective tensor product. After all of this forework, we apply the method of Section 2.2 to prove our main perturbation results.

### 2.3.1 Applying tensor products

Let us begin by reformulating Theorem 2.1. Instead of viewing  $Z$  and  $\mathcal{F}$  as suitable subalgebras of a given Banach algebra  $\mathcal{A}$ , we now construct  $\mathcal{A}$  from the given algebras  $Z$  and  $\mathcal{F}$ .

Let  $Z$  be a commutative Banach algebra with unit element  $e_Z$  and  $\mathcal{F}$  a (not necessarily commutative) complex Banach algebra with unit element  $e_{\mathcal{F}}$ . Let  $Z \otimes \mathcal{F}$  stand for the algebraic tensor product of  $Z$  and  $\mathcal{F}$ . Then by a *reasonable cross norm* ([57], Chapter VIII) we mean a norm on  $Z \otimes \mathcal{F}$  having the following properties:

1.  $\|z \otimes f\| = \|z\| \|f\|$  for  $z \in Z$  and  $f \in \mathcal{F}$ .
2. For every  $z^* \in Z^*$  and  $f^* \in \mathcal{F}^*$  (the duals of  $Z$  and  $\mathcal{F}$  as Banach **spaces**), we have

$$|(x^* \otimes f^*)(a)| \leq \|x^*\| \|f^*\| \|a\|, \quad a \in Z \otimes \mathcal{F}.$$

Writing  $\|\cdot\|_{\alpha}$  for a reasonable cross norm on  $Z \otimes \mathcal{F}$ , we denote its completion by  $Z \otimes_{\alpha} \mathcal{F}$ . It is easily seen that

$$\|a\|_{\varepsilon} \leq \|a\|_{\alpha} \leq \|a\|_{\pi}, \quad a \in Z \otimes \mathcal{F},$$

where

$$\|a\|_\varepsilon = \sup_{\substack{z^* \in Z^*, \|z^*\|=1 \\ f^* \in \mathcal{F}^*, \|f^*\|=1}} |(z^* \otimes f^*)(a)|, \quad a \in Z \otimes \mathcal{F},$$

is the *injective tensor norm* and

$$\|a\|_\pi = \inf \left\{ \sum_j \|z_j\| \|f_j\| : a = \sum_j (z_j \otimes f_j) \right\}, \quad a \in Z \otimes \mathcal{F},$$

is the *projective tensor norm*. Obviously,  $z \mapsto (z \otimes e_{\mathcal{F}})$  and  $f \mapsto (e_Z \otimes f)$  are isometric imbeddings of  $Z$  and  $\mathcal{F}$  into  $Z \otimes \mathcal{F}$ , respectively, provided  $\|e_Z\| = \|e_{\mathcal{F}}\| = 1$ . It is clear that, for any reasonable cross norm  $\|\cdot\|_\alpha$ ,  $\mathcal{A} \stackrel{\text{def}}{=} Z \otimes_\alpha \mathcal{F}$  is a Banach algebra with unit element  $e \stackrel{\text{def}}{=} e_Z \otimes e_{\mathcal{F}} \in Z \cap \mathcal{F}$  (with  $Z \cap \mathcal{F}$  viewed as a subalgebra of  $\mathcal{A}$ ). We shall define reasonable cross norms on the algebraic tensor product of two arbitrary Banach spaces (rather than Banach algebras) in the same way.

Let  $\Phi_\varphi$  be the multiplicative projection associated with the multiplicative functional  $\varphi$  on  $Z$ , initially defined on  $Z \otimes \mathcal{F}$  by (2.1). Then for  $a = \sum_j (z_j \otimes f_j)$  we have

$$\begin{aligned} \|\Phi_\varphi(a)\|_{\mathcal{F}} &= \left\| \sum_j \varphi(z_j) f_j \right\|_{\mathcal{F}} = \sup_{f^* \in \mathcal{F}^*, \|f^*\|=1} \left| \sum_j \varphi(z_j) \langle f_j, f^* \rangle \right| \\ &\leq \sup_{\substack{z^* \in Z^*, \|z^*\|=1 \\ f^* \in \mathcal{F}^*, \|f^*\|=1}} \left| \sum_j \langle z_j, z^* \rangle \langle f_j, f^* \rangle \right| \\ &= \sup_{\substack{z^* \in Z^*, \|z^*\|=1 \\ f^* \in \mathcal{F}^*, \|f^*\|=1}} |(z^* \otimes f^*)(a)| = \|a\|_\varepsilon \leq \|a\|_\alpha, \end{aligned} \quad (2.8)$$

which implies that  $\Phi_\varphi$  has a unique bounded continuation to  $Z \otimes_\alpha \mathcal{F}$ . Note that, at the inequality, we have taken linear functionals on  $Z$  with unit norm to replace multiplicative linear functionals on  $Z$ .

We can now apply Theorem 2.1 and prove

**Theorem 2.7 (Allan-Bochner-Phillips).** *Let  $Z$  be a commutative Banach algebra with unit element,  $\mathcal{F}$  a Banach algebra with unit element, and  $\|\cdot\|_\alpha$  a reasonable cross norm on  $Z \otimes \mathcal{F}$ . Then  $a$  is invertible in  $Z \otimes_\alpha \mathcal{F}$  if and only if  $\Phi(a)$  is invertible in  $\mathcal{F}$  for every multiplicative projection  $\Phi$ .*

### 2.3.2 Bochner vs. Pettis integration

For later use, primarily in the next subsection as well as in Section 7.2, we introduce both the Bochner and the Pettis integral of a function  $f$  from a set  $E$  equipped with a countably additive positive measure  $\mu$  into a complex Banach space  $Y$  and

compile its major properties. For details we refer the reader to [57, Chapter II] and [59, Chapter III].

**1. Bochner integration.** Let  $(E, \mu)$  be a measure space,  $\Sigma$  its underlying  $\sigma$ -algebra,<sup>1</sup> and  $Y$  a complex Banach space. Then  $f : E \rightarrow Y$  is called *simple* if  $F = \{f(t) : t \in E\}$  is a finite set and, for each  $y \in F$ ,  $\{t \in E : f(t) = y\} \in \Sigma$  and  $\mu(\{t \in E : f(t) = y\})$  is finite. A function  $f : E \rightarrow Y$  is called *strongly  $\mu$ -measurable* if there exists a sequence of simple functions  $\{f_n\}_{n=1}^\infty$  on  $E$  such that  $\|f_n(t) - f(t)\|_Y \rightarrow 0$  for  $\mu$ -almost every  $t \in E$ . Then strongly  $\mu$ -measurable functions remain strongly  $\mu$ -measurable under sums, scalar multiples, and pointwise (almost everywhere) strong limits. A strongly  $\mu$ -measurable function  $f : E \rightarrow Y$  is called *Bochner integrable* if there exists a sequence of simple functions  $\{f_n\}_{n=1}^\infty$  such that

$$\lim_{n \rightarrow \infty} \int_E \|f_n(t) - f(t)\|_Y d\mu = 0.$$

We then define the Bochner integral  $\int_E f d\mu$  as the limit of the naturally defined integrals  $\int_E f_n d\mu$  of simple functions. It is easily shown [57, Theorem II 2.2] that a strongly  $\mu$ -measurable function is Bochner integrable if and only if  $\|f(\cdot)\|_Y \in L^1(E, d\mu)$ .

Bochner integrals have many of the usual properties of the Lebesgue integrals of scalar functions, such as the Dominated Convergence Theorem which will be applied in the remainder of the book without further ado. It reads as follows [59, Corollary III.16]: *Let  $\{f_n\}_{n=1}^\infty$  be a sequence of  $\mu$ -measurable functions  $f_n : E \rightarrow X$  such that  $\|f_n(t) - f(t)\|_X \rightarrow 0$  and, for some  $g \in L^1(E, d\mu)$ ,  $\|f_n(t)\|_X \leq g(t)$  for  $\mu$ -almost every  $t \in E$ . Then  $f : E \rightarrow X$  is Bochner integrable and*

$$\lim_{n \rightarrow \infty} \int_E f_n(t) d\mu(t) = \int_E f(t) d\mu(t).$$

Also, Hille's Lemma 1.2 can be stated in terms of Bochner integrals: *If  $S(X \rightarrow X)$  is a closed linear operator,  $f : E \rightarrow X$  takes  $\mu$ -almost all of its values in  $\mathcal{D}(S)$ , and both  $f$  and  $Sf$  are Bochner integrable, then  $\int_E f d\mu \in \mathcal{D}(S)$  and  $S \int_E f d\mu = \int_E Sf d\mu$ .*

**2. Pettis integration.** A function  $f : E \rightarrow Y$  is called *weakly  $\mu$ -measurable* if the scalar functions  $\langle f(\cdot), y^* \rangle$  are measurable for every  $y^* \in Y^*$ . According to Pettis' Measurability Theorem [57, Theorem II 1.2],  $f$  is strongly  $\mu$ -measurable iff  $f$  is weakly  $\mu$ -measurable and there exists  $E_0 \in \Sigma$  with  $\mu(E_0) = 0$  such that  $f(E \setminus E_0)$  is a (norm) separable subset of  $Y$ . If  $f : E \rightarrow Y$  has the additional property that  $\langle f(\cdot), y^* \rangle \in L^1(E, d\mu)$  for each  $y^* \in Y^*$ , then the linear functional  $y^* \mapsto \int_E \langle f(t), y^* \rangle d\mu$  is continuous and hence represents an element of the second dual space  $Y^{**}$ , which we conveniently denote by  $\int_E f d\mu$ . However, if  $\int_E f \chi_F d\mu$

<sup>1</sup>In other words,  $\mu : \Sigma \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  is a countably additive function defined on the  $\sigma$ -algebra  $\Sigma$  of subsets of  $E$ .

belongs to  $Y$  (rather than to  $Y^{**}$ ) for every  $F \in \Sigma$ , then  $f$  is called *Pettis integrable* and its Pettis integral  $(P) \int_E f d\mu$  is defined as follows:

$$\left\langle (P) \int_E f d\mu, y^* \right\rangle = \int_E \langle f(t), y^* \rangle d\mu.$$

According to [57, Theorem II 3.5], for each Pettis integrable function  $f : E \rightarrow Y$  the map

$$F \mapsto (P) \int_E f \chi_F d\mu, \quad F \in \Sigma, \quad (2.9)$$

is a countably additive  $\mu$ -continuous vector measure on  $\Sigma$ . Indeed, its weak countable additivity is obvious. However, the Orlicz-Pettis Theorem [57, Corollary I 4.4] implies that weakly countably additive vector measures on a  $\sigma$ -algebra are (norm) countably additive.

We have the following result [57].

**Proposition 2.8.** *Let  $\mu$  be a  $\sigma$ -finite measure on  $E$  and  $Y$  a complex Banach space. Then the following statements are true:*

1. *The complex Banach space  $L^1(E, Y)$  of all Bochner integrable functions  $f : E \rightarrow Y$  satisfies*

$$L^1(E, Y) = L^1(E, d\mu) \otimes_{\pi} Y.$$

2. *The closure of the complex vector space of all Pettis integrable functions  $f : E \rightarrow Y$  with respect to the norm*

$$\|f\| = \sup_{\|y^*\|=1} \int_E |\langle f(t), y^* \rangle| d\mu \quad (2.10)$$

*coincides with the injective tensor product  $L^1(E, d\mu) \otimes_{\varepsilon} Y$ .*

*Proof.* The first part is well known [57, Example VIII 1.10], also if the measure is not finite. The second part appears as [57, Theorem VIII 1.5] if the measure is finite. If the measure  $\mu$  is  $\sigma$ -finite and  $\{E_n\}_{n=1}^{\infty}$  is an increasing sequence of subsets of  $E$  of finite  $\mu$ -measure with union  $E$ , we use the norm countable additivity of (2.9) to prove that, for Pettis integrable  $f : E \rightarrow Y$ ,

$$\lim_{n \rightarrow \infty} \left\| \int_E f d\mu - \int_{E_n} f d\mu \right\|_Y = 0 \quad (2.11)$$

to approximate  $f$  by Pettis integrable functions  $f_n$  supported on  $E_n$ . From (2.11) we then have

$$\lim_{n \rightarrow \infty} \sup_{\|y^*\|=1} \left| \int_E \langle f(t), y^* \rangle d\mu - \int_{E_n} \langle f(t), y^* \rangle d\mu \right| = 0.$$

Then we employ the second part of Proposition 2.8 for the finite measure  $\mu_n(F) = \mu(F)$  for  $F \in \Sigma$  and  $F \subset E_n$  to approximate  $f_n$  in the norm (2.10) by a function in the algebraic tensor product  $L^1(E_n, Y) \otimes Y$ .  $\square$

We now generalize Hille's Lemma to the realm of Pettis integration. In the process we shall also prove Lemma 1.2.

**Lemma 2.9.** *Let  $X$  and  $Y$  be complex Banach spaces and let  $S(X \rightarrow Y)$  be a closed and densely defined linear operator. Suppose  $(E, \mu)$  is a measure space and  $f : E \rightarrow X$  is Pettis integrable. If  $f(t) \in \mathcal{D}(S)$  for  $\mu$ -a.e.,  $t \in E$  and  $Sf : E \rightarrow Y$  is Pettis integrable, then  $(P) \int_E f(t) d\mu(t) \in \mathcal{D}(S)$  and*

$$S \left( (P) \int_E f(t) d\mu(t) \right) = (P) \int_E Sf(t) d\mu(t).$$

*Proof.* For  $\phi \in \mathcal{D}(S^*)$  we compute

$$\begin{aligned} \left\langle (P) \int_E f(t) d\mu(t), S^* \phi \right\rangle &= \int_E \langle f(t), S^* \phi \rangle d\mu(t) \\ &= \int_E \langle Sf(t), \phi \rangle d\mu(t) \\ &= \left\langle (P) \int_E Sf(t) d\mu(t), \phi \right\rangle. \end{aligned}$$

Since this calculation can be repeated with  $f\chi_F$  instead of  $f$  for any  $F \in \Sigma$ , we conclude that  $Sf : E \rightarrow Y$  is Pettis integrable. The lemma is then immediate from Proposition 1.1.  $\square$

### 2.3.3 Bounded additive perturbations

Our principal application of Theorem 2.7 has the following form. Let  $\mathcal{B}$  be a complex Banach algebra with unit element  $e_{\mathcal{B}}$  and let  $Z$  stand for the continuous Wiener algebra  $\mathbb{C} + L^1(\mathbb{R})$ . Then  $Z = L^1(E, \mu)$ , where  $E = \mathbb{R} \cup \{\infty\}$ ,  $\mu$  is the Lebesgue measure on  $\mathbb{R}$ , and  $\mu(\{\infty\}) = 1$ . Then according to the first part of Proposition 2.8 we have

$$L^1(E, \mathcal{B}) = Z \otimes_{\pi} \mathcal{B}$$

in the sense of norm isometry. Since the multiplicative linear functionals on  $Z$  are Fourier transform maps (including the Fourier transform evaluation at infinity), we see that  $A$  is invertible in  $L^1(E, \mathcal{B})$  if and only if the values of its Fourier transform (including its value at infinity) are invertible in  $\mathcal{B}$ .

Before proving the most general additive perturbation result on Hilbert spaces, we derive the following useful lemma.

**Lemma 2.10.** *Let  $X$  be a complex Banach space. Suppose  $K : \mathbb{R} \times X \rightarrow X$  is a vector function such that*

- (1)  $K(\cdot, x) \in L^1(\mathbb{R}; X)$  for every  $x \in X$ ,
- (2)  $K(t, \lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 K(t, x_1) + \lambda_2 K(t, x_2)$  for  $\lambda_1, \lambda_2 \in \mathbb{C}$ ,  $x_1, x_2 \in X$ , and a.e.  $t \in \mathbb{R}$ .

Then there exists a unique bounded linear operator  $L$  on  $L^1(\mathbb{R}; X)$  such that

$$(L\psi)(t) = \int_{-\infty}^{\infty} K(t - \tau, \psi(\tau)) d\tau \quad (2.12)$$

for integrable step functions  $\psi$ , while the norm of  $L$  is bounded above by

$$\sup_{\|x\|=1} \int_{-\infty}^{\infty} \|K(t, x)\| dt.$$

*Proof.* Let

$$\psi(t) = \sum_{j=1}^n \chi_{E_j}(t) x_j$$

be a nontrivial Bochner integrable step function, i.e.,  $\{x_1, \dots, x_n\} \subset X \setminus \{0\}$  and  $E_1, \dots, E_n$  are mutually disjoint subsets of  $\mathbb{R}$  of finite Lebesgue measure. Then the integral in (2.12) is well defined as a Bochner integral and equals

$$\begin{aligned} (L\psi)(t) &= \sum_{j=1}^n \int_{-\infty}^{\infty} \chi_{E_j}(\tau) K(t - \tau, x_j) d\tau \\ &= \sum_{j=1}^n \|x_j\| \int_{-\infty}^{\infty} \chi_{E_j}(\tau) K\left(t - \tau, \frac{x_j}{\|x_j\|}\right) d\tau. \end{aligned}$$

Moreover,

$$\begin{aligned} \|L\psi\|_{L^1(\mathbb{R}; X)} &\leq \sum_{j=1}^n \|x_j\| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{E_j}(\tau) \left\| K\left(t - \tau, \frac{x_j}{\|x_j\|}\right) \right\| d\tau dt \\ &= \sum_{j=1}^n \|x_j\| \int_{-\infty}^{\infty} \chi_{E_j}(\tau) \left( \int_{-\infty}^{\infty} \left\| K\left(t - \tau, \frac{x_j}{\|x_j\|}\right) \right\| dt \right) d\tau \\ &\leq \left( \sup_{\|x\|=1} \int_{-\infty}^{\infty} \|K(t, x)\| dt \right) \sum_{j=1}^n m(E_j) \|x_j\| \\ &= \left( \sup_{\|x\|=1} \int_{-\infty}^{\infty} \|K(t, x)\| dt \right) \|\psi\|_{L^1(\mathbb{R}; X)}. \end{aligned}$$

Using the density of the Bochner integrable step functions in  $L^1(\mathbb{R}; X)$  we easily prove that  $L$  extends uniquely to a bounded linear operator on  $L^1(\mathbb{R}; X)$  with the above norm bound.  $\square$

We have not stated that the integral representation (2.12) is valid for any  $\psi \in L^1(\mathbb{R}; X)$ . However, if  $K(t, x) = K(t)x$  for some  $K(t) \in \mathcal{L}(X)$  and  $\|K(t)x\| \leq \varphi(t)\|x\|$  a.e. for some  $\varphi \in L^1(\mathbb{R})$ , then the right-hand side of (2.12) is a Bochner

integral and (2.12) is true for each  $\psi \in L^1(\mathbb{R}; X)$ . In the sequel we shall apply Lemma 2.10 primarily for  $K(t, x) = E(t)x$ , where  $E$  is a strongly continuous bisemigroup on  $X$ .

It is not clear if Lemma 2.10 is valid on  $L^p(\mathbb{R}; X)$  for  $p > 1$ . However, it is valid on  $L^2(\mathbb{R}; X)$  if  $X$  is a complex Hilbert space.

**Lemma 2.11.** *Let  $X$  be a complex Hilbert space and let  $K : \mathbb{R} \times X \rightarrow X$  be a vector function satisfying the hypotheses of Lemma 2.10. Then the linear operator  $L$  defined by (2.12) for each integrable step function  $\psi$  extends to a unique bounded linear operator on  $L^2(\mathbb{R}; X)$  with norm given by*

$$\|L\| = \sup_{\|x\|=1, \lambda \in \mathbb{R}} \|\hat{K}(\lambda, x)\|,$$

where

$$\hat{K}(\lambda, x) = \int_{-\infty}^{\infty} e^{i\lambda t} K(t, x) dt.$$

*Proof.* Suppose  $\mathcal{F}$  is the Fourier transform map on  $L^2(\mathbb{R}; X)$ . Then  $(2\pi)^{-1/2}\mathcal{F}$  is a unitary operator. Moreover, letting  $\psi$ , an integrable step function, be as in the proof of Lemma 2.10, we get for  $\lambda \in \mathbb{R}$ ,

$$(\mathcal{F}L\psi)(\lambda) = \sum_{j=1}^n \|x_j\| \int_{-\infty}^{\infty} \widehat{\chi_{E_j}}(\lambda) \hat{K}\left(\lambda, \frac{x_j}{\|x_j\|}\right) = \hat{K}(\lambda, (\mathcal{F}\psi)(\lambda)).$$

Then the bounded extendability of  $L$  and the exact expression for its norm easily follow with the help of the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}; X) & \xrightarrow{L} & L^2(\mathbb{R}; X) \\ \mathcal{F} \downarrow & & \downarrow \mathcal{F} \\ L^2(\mathbb{R}; X) & \xrightarrow{\lambda \mapsto \hat{K}(\lambda, \cdot)} & L^2(\mathbb{R}; X), \end{array}$$

as claimed. □

We have not stated that the integral representation (2.12) is valid for any  $\psi \in L^2(\mathbb{R}; X)$ . However, if  $K(t, x) = K(t)x$  for some  $K(t) \in \mathcal{L}(X)$  and  $\|K(t)x\| \leq \varphi(t)\|x\|$  a.e. for some  $\varphi \in L^1(\mathbb{R})$ , then the right-hand side of (2.12) is a Bochner integral and (2.12) is true for each  $\psi \in L^1(\mathbb{R}; X)$ . In the sequel we shall apply Lemma 2.10 primarily for  $K(t, x) = E(t)x$ , where  $E$  is a strongly continuous bisemigroup on  $X$ . Only in Subsection 7.2.1 shall we encounter less trivial applications of Lemma 2.10, where we are not going to bother with the precise nature of (2.12) when  $\psi$  is not a step function.

As an immediate consequence of Lemma 2.10 and Theorem 1.7, we prove that bounded perturbations of exponentially dichotomous operators on complex Banach spaces with a sufficiently small norm are themselves exponentially dichotomous.



**Theorem 2.12.** *Let  $S_0$  be an exponentially dichotomous operator on a complex Banach space  $X$ , and let  $\Gamma \in \mathcal{L}(X)$ . Then there exists  $\delta = \delta(S_0) > 0$  such that  $S = S_0 + \Gamma$  is exponentially dichotomous whenever  $\|\Gamma\| < \delta$ .*

*Proof.* To derive Theorem 2.12 from Theorem 1.7 and Lemma 2.10, we remark that for certain  $c, M > 0$ ,

$$\|E(t; S_0)x\|_{\mathcal{L}(X)} \leq M e^{-c|t|}\|x\|, \quad 0 \neq t \in \mathbb{R}, x \in X.$$

Then

$$\int_{-\infty}^{\infty} e^{\varepsilon|t|}\|E(t; S_0)x\| dt \leq \frac{2M\|x\|}{c - \varepsilon}, \quad x \in X,$$

whenever  $0 < \varepsilon < c$ . Then, according to Lemma 2.10, for each  $\delta \in (-c, c)$  the convolution operator

$$(L_\delta \psi)(t) = \int_{-\infty}^{\infty} e^{\delta(t-\tau)} E(t-\tau; S_0) \Gamma \psi(s) ds \quad (2.13)$$

is bounded on  $L^1(\mathbb{R}; X)$  with norm satisfying

$$\|L_\delta\| \leq \sup_{\|x\|=1} \int_{-\infty}^{\infty} e^{\delta t} \|E(t; S_0) \Gamma x\| dt \leq \frac{2cM\|\Gamma\|_{\mathcal{L}(X)}}{c^2 - \delta^2}.$$

Thus if  $\|\Gamma\| < (c/2M)$  and  $0 < \delta < \sqrt{c(c - 2M\|\Gamma\|)}$ , the convolution integral equation

$$F(t; x) - \int_{-\infty}^{\infty} E(t-\tau; S_0) \Gamma F(\tau; x) d\tau = E(t; S_0)x \quad (2.14)$$

has a unique solution such that  $e^{\delta(\cdot)} F(\cdot; x) \in L^1(\mathbb{R}; X)$  for every  $x \in X$ . In particular,  $F(\cdot; x)$  is strongly measurable for each  $x \in X$ .

The norms of the integral equation (2.14) are dominated by those of the scalar integral equation

$$\Phi(t) - M\|\Gamma\| \int_{-\infty}^{\infty} e^{-c|t-\tau|} \Phi(\tau) d\tau = M\|x\|e^{-c|t|},$$

which has

$$\Phi(t) = \frac{cM\|x\|}{\sqrt{c(c - 2M\|\Gamma\|)}} e^{-t\sqrt{c(c - 2M\|\Gamma\|)}}$$

as its unique solution whenever  $\|\Gamma\| < (c/2M)$ . In that case

$$\|F(t; x)\| \leq \frac{cM\|x\|}{\sqrt{c(c - 2M\|\Gamma\|)}} e^{-t\sqrt{c(c - 2M\|\Gamma\|)}}$$

for a.e.  $t \in \mathbb{R}$  and each  $x \in X$ . Moreover,

$$(\lambda - S)^{-1}x = \int_{-\infty}^{\infty} e^{-\lambda t} F(t; x) dt, \quad |\operatorname{Re} \lambda| < \sqrt{c(c - 2M\|\Gamma\|)}.$$

Then Theorem 1.7 implies that  $S$  is exponentially dichotomous.  $\square$

We now derive the most general perturbation result on complex Hilbert spaces.

**Theorem 2.13.** *Let  $S_0$  be an exponentially dichotomous operator on the complex Hilbert space  $X$ ,  $\Gamma \in \mathcal{L}(X)$ , and  $S = S_0 + \Gamma$ . Suppose*

$$C_h = \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\} \subset \rho(S)$$

*for some  $h > 0$ . Moreover, suppose  $(\lambda - S)^{-1}$  is bounded on  $C_h$ . Then  $S$  is exponentially dichotomous.*

*Proof.* For some  $\varepsilon \in (0, h]$ ,  $e^{\varepsilon|\cdot|}E(\cdot; S_0)x : \mathbb{R} \rightarrow X$  is Pettis integrable for each  $x \in X$ . Thus, by Lemma 2.11, for each  $\delta \in (-\varepsilon, \varepsilon)$  the convolution operator given by (2.13) is bounded on  $L^2(\mathbb{R}; X)$  with norm bounded above by

$$\sup_{\lambda \in i\mathbb{R}} \|W(\lambda - \delta) - I_X\|_{\mathcal{L}(X)}, \quad (2.15)$$

where

$$W(\lambda) = I_X - (\lambda - S_0)^{-1}\Gamma, \quad |\operatorname{Re} \lambda| < \varepsilon.$$

This follows easily using the unitarity of  $(2\pi)^{-1/2}\mathcal{F}_X$  on  $L^2(\mathbb{R}; X)$ , where  $\mathcal{F}_X$  is the Fourier transform, given that  $X$  is a complex Hilbert space. Therefore, for each  $\delta \in (-\varepsilon, \varepsilon)$  the convolution operator  $L_0$  is bounded on  $L^2(\mathbb{R}, e^{2\delta t}dt; X)$  with norm bounded above by (2.15). Moreover, since  $W(\lambda)^{-1} = I_X + (\lambda - S)^{-1}\Gamma$  for  $|\operatorname{Re} \lambda|$  sufficiently small, we have, as a result of the second hypothesis on  $S$ ,

$$M_\delta \stackrel{\text{def}}{=} \sup_{\operatorname{Re} \lambda = -\delta} \|W(\lambda)^{-1}\| < \infty, \quad \delta \in [-h, h]. \quad (2.16)$$

Now consider the vector-valued convolution equation (2.14), where  $x \in H$ . Then for  $\delta \in (-\varepsilon, \varepsilon)$  any solution  $F(\cdot; x)$  belonging to  $L^2(\mathbb{R}, e^{2\delta t}dt; X)$  satisfies

$$\int_{-\infty}^{\infty} e^{-\lambda t} F(t; x) dt = W(\lambda)^{-1}(\lambda - S_0)^{-1}x = (\lambda - S)^{-1}x, \quad (2.17)$$

where  $|\operatorname{Re} \lambda| < \varepsilon$ . Since by (2.15)

$$\begin{aligned} \int_{-\infty}^{\infty} \|F(t; x)\|^2 e^{2\delta t} dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \|(-\delta + i\mu - S)^{-1}x\| d\mu \\ &\leq \frac{(M_\delta)^2}{2\pi} \int_{-\infty}^{\infty} \|(-\delta + i\mu - S_0)^{-1}x\| d\mu \\ &\leq (M_\delta)^2 \int_{-\infty}^{\infty} \|E(t; S_0)x\|^2 e^{2\delta t} dt < \infty, \end{aligned}$$

where  $x \in X$ , then for each  $\delta \in (-\varepsilon, \varepsilon)$  and every  $x \in X$  there exists a unique solution  $F(\cdot; x)$  of (2.17) in  $L^2(\mathbb{R}, e^{2\delta t}dt; X)$ . In particular,  $F(\cdot; x)$  is strongly measurable for each  $x \in X$ .

Now note that  $F(\cdot; x) \in L^1(\mathbb{R}, e^{2\varepsilon t} dt; X)$  for every  $\gamma \in (0, \delta)$  and each  $x \in X$ , while

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\gamma|t|} \|F(t; x)\| dt &\leq \left[ \int_{-\infty}^{\infty} e^{-2(\delta-\gamma)|t|} dt \right]^{1/2} \left[ \int_{-\infty}^{\infty} e^{2\delta|t|} \|F(t; x)\|^2 dt \right]^{1/2} \\ &\leq \frac{\max(M_\delta, M_{-\delta})}{\sqrt{\delta-\gamma}} \left[ \int_{-\infty}^{\infty} e^{2\delta|t|} \|E(t; S_0)x\|^2 dt \right]^{1/2} \\ &\leq \frac{\max(M_\delta, M_{-\delta})}{\sqrt{\delta-\gamma}} \frac{M}{\sqrt{c-\delta}} \|x\| \stackrel{\text{def}}{=} \mu \|x\|. \end{aligned}$$

Since

$$F(t; x) = E(t; S_0)x - \int_{-\infty}^{\infty} F(t - \tau; \Gamma E(\tau; S_0))x d\tau,$$

we can use the iteration argument in the second part of the proof of Theorem 2.12 and prove that

$$\|F(t; x)\| \leq \frac{cM\|x\|}{\sqrt{c(c-2\mu\|\Gamma\|)}} e^{-|t|\sqrt{c(c-2\mu\|\Gamma\|)}}, \quad t \in \mathbb{R} \text{ a.e.}$$

Then Theorem 1.7 implies that  $S$  is exponentially dichotomous. □

## Chapter 3

# Abstract Cauchy Problems

If  $S(X \rightarrow X)$  is the infinitesimal generator of a strongly continuous semigroup on a complex Banach space, the literature about such an  $S$  abounds with results on the existence of a unique solution of the Cauchy problem

$$\begin{cases} u'(t) = Su(t) + f(t), & t \in \mathbb{R}^+, \\ u(0^+) = x_0. \end{cases}$$

In this chapter we study the existence and uniqueness of classical, weak, and mild solutions to the Cauchy problem

$$\begin{cases} u'(t) = Su(t) + f(t), & 0 \neq t \in \mathbb{R}, \\ u(0^+) - u(0^-) = x_0, \end{cases}$$

where now  $S$  is an exponentially dichotomous operator. Furthermore, we characterize exponentially dichotomous operators as to their ability to lead to uniquely solvable Cauchy problems.

### 3.1 The abstract Cauchy problem

In this section we discuss the existence and uniqueness of classical, weak, and mild solutions of the inhomogeneous Cauchy problem generated by a closed and densely defined linear operator  $S(X \rightarrow X)$ . First we consider the case in which  $S$  is the infinitesimal generator of a strongly continuous semigroup. Next, we assume  $S$  to be exponentially dichotomous.

#### 3.1.1 Involving semigroup generators

If  $S$  is the infinitesimal generator of a strongly continuous semigroup on a complex Banach space  $X$ , then there is extensive theory on abstract Cauchy problems of

the type

$$u'(t) = Su(t) + f(t), \quad t \in (0, T), \quad (3.1a)$$

$$u(0^+) = x_0, \quad (3.1b)$$

where  $T > 0$ . From the formal point of view, we can write (3.1) in the form

$$u(t) = e^{tS}x_0 + \int_0^t e^{(t-\tau)S}f(\tau) d\tau, \quad t \in [0, T], \quad (3.2)$$

which is the so-called *variation of constants formula*. If  $f \in L^1((0, T); X)$ , we call  $u \in C([0, T]; X)$  a *mild solution* of (3.1) if it is given by (3.2). A vector function  $u : [0, T] \rightarrow X$  is called a *classical solution* of (3.1) if it is continuous on  $[0, T]$ , is continuously differentiable on  $(0, T)$ , its values for  $t \in (0, T)$  belong to  $\mathcal{D}(S)$ , and (3.1) is satisfied for every  $t \in (0, T)$ . If  $f \in L^1([0, T]; X)$ , then (3.1) has at most one classical solution and this solution is given by (3.2). Finally, by a *weak solution* of (3.1) we mean a vector function  $u \in C([0, T]; X)$  such that (3.1b) holds and for every  $\varphi \in \mathcal{D}(S^*)$  the function  $\langle u(\cdot), \varphi \rangle$  is absolutely continuous on  $[0, T]$  and

$$\frac{d}{dt} \langle u(t), \varphi \rangle = \langle u(t), S^* \varphi \rangle + \langle f(t), \varphi \rangle \quad \text{a.e. on } [0, T].$$

In [127] the connections between classical and mild solutions and the special situation of having an analytic semigroup generator are discussed in detail. For weak solutions we refer to [11].

In fact, we have the following result ([11] for the weak case and [127, 103] for the classical case).

**Theorem 3.1.** *Let  $S$  be the infinitesimal generator on a strongly continuous semigroup on the complex Banach space  $X$  and let  $f \in L^1((0, T); X)$ . Then for every  $x_0 \in X$  there exists a unique weak solution of the Cauchy problem (3.1) and this solution is mild. If in addition  $x_0 \in \mathcal{D}(S)$  and  $f : [0, T] \rightarrow X$  is strongly continuously differentiable, then this solution is classical.*

If  $S$  generates an analytic semigroup on  $X$ , then it is sufficient to require  $x_0 \in X$  and  $\|f(t) - f(s)\|_X \leq L|t - s|^\varepsilon$  for some  $\varepsilon \in (0, 1]$  and all  $t, s \in [0, T]$ , for the Cauchy problem (3.1) to have a classical solution [103, Theorem IX 1.27]. More detailed information on the Cauchy problem (3.1) with  $S$  the infinitesimal generator of an analytic semigroup can be found in [127, 118].

### 3.1.2 Involving exponentially dichotomous operators

If  $S$  is an exponentially dichotomous operator on a complex Banach space  $X$ , the natural generalization of the variation of constants formula (3.2) is as follows:

$$u(t) = E(t; S)x_0 + \int_{-T_2}^{T_1} E(t - \tau; S)f(\tau) d\tau, \quad t \in [-T_2, T_1], \quad (3.3)$$

where  $T_1, T_2 > 0$  and  $f \in L^1((-T_2, T_1); X)$ . By differentiating (3.3) with total disregard of differentiability issues, we obtain the natural generalization of the abstract Cauchy problem (3.1), namely

$$u'(t) = Su(t) + f(t), \quad 0 \neq t \in (-T_2, T_1), \quad (3.4a)$$

$$u(0^+) - u(0^-) = x_0, \quad (3.4b)$$

where  $T_1, T_2 > 0$ .

As in the semigroup case, we now define various types of solutions of (3.4). Given  $f \in L^1((-T_2, T_1); X)$ , by a *mild solution* of (3.4) we mean the vector function

$$u \in C([-T_2, 0]; X) \dot{+} C([0, T_1]; X) \quad (3.5)$$

given by (3.3). In particular, if  $u$  is a mild solution of (3.4), then the strong one-sided limits  $u(0^\pm)$  exist.

The following lemma implies the existence of a mild solution of the Cauchy problem (3.4) if  $x_0 \in X$  and  $f \in L^1((-T_2, T_1); X)$ .

**Lemma 3.2.** *Let  $S$  be an exponentially dichotomous operator on the complex Banach space  $X$  and let  $f \in L^1((-T_2, T_1); X)$ . Then the vector function defined by*

$$v(t) = \int_{-T_2}^{T_1} E(t - \tau; S) f(\tau) d\tau \quad (3.6)$$

*belongs to  $L^1(\mathbb{R}; X) \cap BC(\mathbb{R}; X)$ . Moreover, for every  $x_0 \in X$  the vector function  $u$  given by (3.3) is a mild solution of the Cauchy problem (3.4).*

*Proof.* Applying Lemma 2.10 we easily prove that the vector function  $v$  defined by (3.6) belongs to  $L^1(\mathbb{R}; X)$ . Moreover, using the existence of  $\varepsilon > 0$  such that  $e^{\varepsilon|\cdot|}E(\cdot; S)x \in L^1(\mathbb{R}; X)$  for any  $x \in X$ , we can actually prove that  $e^{\varepsilon|\cdot|}v \in L^1(\mathbb{R}; X)$ . To prove that  $v \in BC(\mathbb{R}; X)$ , we first assume that  $f \in C([-T_2, T_1]; X)$  and apply the Theorem of Dominated Convergence to prove the strong continuity of  $v$ ; its boundedness follows trivially. In general, if  $f \in L^1((-T_2, T_1); X)$ , we approximate  $f$  in the norm of  $L^1((-T_2, T_1); X)$  by a sequence  $\{f_n\}_{n=1}^\infty$  of vector functions in  $C([-T_2, T_1]; X)$  and define

$$v_n(t) = \int_{-T_2}^{T_1} E(t - \tau; S) f_n(\tau) d\tau. \quad (3.7)$$

Then  $v_n \in L^1(\mathbb{R}; X) \cap BC(\mathbb{R}; X)$  for  $n \in \mathbb{N}$ . Moreover,

$$\|v_n(t) - v_m(t)\|_X \leq \left( \sup_{0 \neq \tau \in \mathbb{R}} \|E(\tau; S)\| \right) \int_{-T_2}^{T_1} \|f_n(t) - f_m(t)\|_X dt, \quad (3.8)$$

so that  $\{v_n(t)\}_{n=1}^\infty$  converges in the norm of  $X$ , uniformly in  $t \in \mathbb{R}$ . Lemma 2.10 implies that

$$\|v_n - v_m\|_{L^1(\mathbb{R}; X)} \leq \left( \sup_{\|x\|=1} \|E(\cdot; S)x\|_{L^1(\mathbb{R}; X)} \right) \|f_n - f_m\|_{L^1((-T_2, T_1); X)},$$

so that  $\{v_n\}_{n=1}^\infty$  converges to  $v$  in the norm of  $L^1(\mathbb{R}; X)$ . Hence the vector function  $v \in BC(\mathbb{R}; X) \cap L^1(\mathbb{R}; X)$ , as claimed.

It now follows that the vector function  $u$  given by (3.3) satisfies  $u(t) = E(t; S)x_0 + v(t)$  for  $0 \neq t \in \mathbb{R}$  as well as (3.4b). Therefore  $u$  is a mild solution of the Cauchy problem (3.4).  $\square$

A vector function  $u$  satisfying (3.5) is called a *classical solution* of (3.4) if it is continuously differentiable on  $(-T_2, 0) \cup (0, T_1)$ ,  $u(t) \in \mathcal{D}(S)$  for  $t \in (-T_2, 0) \cup (0, T_1)$ , and (3.4) is satisfied. We call a vector function  $u$  satisfying (3.5) (and hence having strong one-sided limits  $u(0^\pm)$ ) a *weak solution* of (3.4) if for every  $\phi \in \mathcal{D}(S^*)$  the function  $\langle u(\cdot), \phi \rangle$  is absolutely continuous on  $[-T_2, T_1]$ , the jump condition (3.4b) holds, and

$$\frac{d}{dt} \langle u(t), \phi \rangle = \langle u(t), S^* \phi \rangle + \langle f(t), \phi \rangle \quad \text{a.e. on } [-T_2, T_1]. \quad (3.9)$$

Evidently, if  $T_1 = T$ ,  $S$  is an infinitesimal generator of a strongly continuous exponentially decaying semigroup on  $X$ , and  $u(t) = 0$  for  $-T_2 < t < 0$ , we obtain the various definitions of solutions in the semigroup case from those in the bisemigroup case.

The next result provides necessary conditions for the existence (but not for the uniqueness) of a weak solution and a classical solution of (3.4), respectively.

**Theorem 3.3.** *Suppose  $S$  is an exponentially dichotomous operator on the complex Banach space  $X$  and let  $f \in L^1((-T_2, T_1); X)$ . Then for every  $x_0 \in X$  the Cauchy problem (3.4) has a weak solution and this solution coincides with the mild solution (3.3). If in addition  $x_0 \in \mathcal{D}(S)$  and  $f$  is strongly continuously differentiable in  $[-T_2, T_1]$ , then this solution is classical.*

*Proof.* Let  $S$  be exponentially dichotomous on  $X$ , let  $x_0 \in X$ , and suppose  $f \in L^1((-T_2, T_1); X)$ . Then the vector function  $u$  given by (3.3) is a mild solution of the Cauchy problem (3.4). Letting  $\varphi \in \mathcal{D}(S^*)$  act on (3.3) we have

$$\langle u(t), \phi \rangle = \langle x_0, E(t; S)^* \phi \rangle + \int_{-T_2}^{T_1} \langle E(t - \tau; S) f(\tau), \phi \rangle d\tau, \quad (3.10)$$

where  $0 \neq t \in \mathbb{R}$ . Since  $\phi \in \mathcal{D}(S^*)$ , the first term in the right-hand side of (3.10) is differentiable for  $0 \neq t \in \mathbb{R}$  and

$$\frac{d}{dt} \langle x_0, E(t; S)^* \phi \rangle = \langle x_0, S^* E(t; S)^* \phi \rangle = \langle u(t), S^* \phi \rangle, \quad 0 \neq t \in \mathbb{R}.$$

Next, if  $f \in C([-T_2, T_1]; X)$  and  $f(-T_2) = f(T_1) = 0$ , then the vector functions

$$(t, \tau) \mapsto \langle f(\tau), E(t - \tau; S)^* \phi \rangle, \quad (t, \tau) \mapsto \langle f(\tau), E(t - \tau; S)^* S^* \phi \rangle$$

are continuous on  $\{(t, \tau) \in \mathbb{R}^2 : \pm(t - \tau) \geq 0\}$ , while

$$\frac{\partial}{\partial t} \langle f(\tau), E(t - \tau; S)^* \phi \rangle = \langle f(\tau), E(t - \tau; S)^* S^* \phi \rangle$$

on  $\{(t, \tau) \in \mathbb{R}^2 : \pm(t - \tau) > 0\}$ . Considering the cases  $t \leq -T_2$ ,  $t \in (-T_2, T_1)$ , and  $t \geq T_1$  separately and writing the second term in the right-hand side of (3.10) as the sum of the two integrals over  $[-T_2, t]$  and  $[t, T_1]$  when we are dealing with the second case, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{-T_2}^{T_1} \langle E(t - \tau; S) f(\tau), \phi \rangle d\tau &= \langle f(t), \phi \rangle + \int_{-T_2}^{T_1} \langle f(\tau), E(t - \tau; S)^* S^* \phi \rangle d\tau \\ &= \langle f(t), \phi \rangle + \int_{-T_2}^{T_1} \langle E(t - \tau; S) f(\tau), S^* \phi \rangle d\tau, \end{aligned}$$

which implies that the vector function  $u$  in (3.3) is a weak solution of (3.4).

Now suppose  $f \in L^1((-T_2, T_1); X)$ . Then we approximate  $f$  in the norm of  $L^1((-T_2, T_1); X)$  by a sequence  $\{f_n\}_{n=1}^\infty$  of vector functions on  $[-T_2, T_1]$  satisfying  $f_n(-T_2) = f_n(T_1) = 0$ . Then the vector functions  $v_n$  defined by (3.7) satisfy (3.8), so that  $v_n \rightarrow v$  in the norm of  $BC(\mathbb{R}; X)$ . Now put

$$u_n(t) = E(t; S)x_0 + v_n(t), \quad 0 \neq t \in \mathbb{R}.$$

Passing to the limit in the two equalities

$$\begin{aligned} \langle v_n(t), \phi \rangle &= \int_{-T_2}^{T_1} \langle E(t - \tau; S) f_n(\tau), \phi \rangle d\tau, \\ \langle v_n(t), S^* \phi \rangle &= \int_{-T_2}^{T_1} \langle E(t - \tau; S) f_n(\tau), S^* \phi \rangle d\tau, \end{aligned}$$

we see that  $\langle u(t), \phi \rangle$  is differentiable for a.e.  $0 \neq t \in \mathbb{R}$  and satisfies (3.9). Thus  $u$  is a weak solution of the Cauchy problem (3.4).

Now let  $S$  be exponentially dichotomous on  $X$ , let  $x_0 \in \mathcal{D}(S)$ , and let  $f$  be continuously differentiable on  $[-T_2, T_1]$ . Put

$$\begin{aligned} u(t) &= E(t; S)x_0 + \int_{t-T_1}^{t+T_2} E(t - \tau; S) f(\tau) d\tau \\ &= E(t; S)x_0 + \int_{-T_2}^0 E(\tau; S) f(t - \tau) d\tau + \int_0^{T_1} E(\tau; S) f(t - \tau) d\tau, \end{aligned}$$



where  $0 \neq t \in \mathbb{R}$ . Then  $u$  is strongly continuous in  $0 \neq t \in \mathbb{R}$ ,  $u$  has a strong jump discontinuity in  $t = 0$  (i.e.,  $u(0^+) - u(0^-) = x_0$ ), and  $u - E(\cdot; S)x_0$  is strongly differentiable with derivative

$$\frac{d}{dt}(u(t) - E(t; S)x_0) = \int_{-T_2}^{T_1} E(\tau; S)f'(t - \tau) d\tau.$$

Thus  $u$  is a classical solution of (3.4). □

In the semigroup case, weak solutions are unique [see Theorem 3.1]. Indeed, using an argument given in [11], letting  $u, \tilde{u}$  be two weak solutions of the Cauchy problem (3.1), put  $w = u - \tilde{u}$ . Then for  $\phi \in \mathcal{D}(S^*)$  we have

$$\frac{d}{dt}\langle w(t), \phi \rangle = \langle w(t), S^*\phi \rangle, \quad 0 < t < T,$$

while  $w(0^+) = 0$ . Thus

$$\langle w(t), \phi \rangle = \int_0^t \langle w(\tau), S^*\phi \rangle d\tau = \left\langle \int_0^t w(\tau) d\tau, S^*\phi \right\rangle,$$

where the second equality follows from  $w \in C([0, T]; X)$ . Hence, by Proposition 1.1,  $\int_0^t w(\tau) d\tau \in \mathcal{D}(S)$  and  $S \int_0^t w(\tau) d\tau = w(t)$ . Consequently, if we assume  $S$  to generate a semigroup, we get  $w(t) \equiv 0$  from  $w(0^+) = 0$  and uniqueness follows.

To deal with the (non)uniqueness of weak solutions in the bisemigroup case, let  $u, \tilde{u}$  be two weak solutions of (3.4), and let  $w = u - \tilde{u}$ . Then  $w \in C([-T_2, T_1]; X)$ ,  $\langle w(\cdot), \phi \rangle$  is absolutely continuous, and for every  $\phi \in \mathcal{D}(S^*)$  we have

$$\frac{d}{dt}\langle w(t), \phi \rangle = \langle w(t), S^*\phi \rangle, \quad t \in [-T_2, T_1] \text{ a.e.} \quad (3.11)$$

Then

$$\langle w(t), \phi \rangle - \langle w(0), \phi \rangle = \int_0^t \langle w(\tau), S^*\phi \rangle d\tau = \left\langle \int_0^t w(\tau) d\tau, S^*\phi \right\rangle, \quad \phi \in \mathcal{D}(S^*),$$

where the second implication follows from the strong continuity of  $w$  (with strong jump discontinuity in  $t = 0$ ). Simple examples in finite-dimensional spaces  $X$  suffice to create examples of nonunique weak solvability of the Cauchy problem (3.4), where  $w(0)$  is a parameter in the solution.

To make sense out of uniqueness results, we should replace (3.4) by the Cauchy problem

$$u'(t) = Su(t) + f(t), \quad 0 \neq t \in \mathbb{R}, \quad (3.12a)$$

$$u(0^+) - u(0^-) = x_0, \quad (3.12b)$$

where  $x_0 \in X$  and  $f \in L^1(\mathbb{R}; X)$ . If  $u, \tilde{u} \in L^1(\mathbb{R}; X)$  both satisfy (3.12), then  $w = u - \tilde{u} \in L^1(\mathbb{R}; X)$  is strongly continuous in  $t \in \mathbb{R}$  and satisfies (3.11) (with  $t \in \mathbb{R}$ ) for each  $\phi \in \mathcal{D}(S^*)$ . Applying the Laplace transform

$$\mathcal{L}[w](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} w(t) dt,$$

we obtain

$$\lambda \langle \mathcal{L}[w](\lambda), \phi \rangle = \langle \mathcal{L}[w](\lambda), S^* \phi \rangle, \quad \phi \in \mathcal{D}(S^*), \quad |\operatorname{Re} \lambda| \leq \varepsilon,$$

for sufficiently small  $\varepsilon > 0$ . Applying Proposition 1.1 we obtain  $\mathcal{L}[w](\lambda) \in \mathcal{D}(S)$  and  $S\mathcal{L}[w](\lambda) = \lambda\mathcal{L}[w](\lambda)$  for sufficiently small  $|\operatorname{Re} \lambda|$ , which implies  $\mathcal{L}[w](\lambda) \equiv 0$  and therefore  $\langle \mathcal{L}[w](\lambda), \phi \rangle \equiv 0$  for each  $\phi \in X^*$ . As a result of the uniqueness theorem for Laplace transforms [164],  $w = 0$  and uniqueness follows. Thus, if  $f \in L^1(\mathbb{R}; X)$ , Eqs. (3.12) have the unique weak (and hence mild) solution

$$u(t) = E(t; S)x_0 + \int_{-\infty}^{\infty} E(t - \tau; S)f(\tau) d\tau, \quad 0 \neq t \in \mathbb{R}, \quad (3.13)$$

in  $L^1(\mathbb{R}; X)$ , which is a classical solution if  $x_0 \in \mathcal{D}(S)$  and  $f$  is also bounded and strongly continuously differentiable with bounded derivative.

### 3.1.3 Involving analytic bisemigroups

If  $S$  is the infinitesimal generator of an analytic semigroup, Theorem 3.1 can be sharpened substantially [127, Sec. 4.3]. For instance, if  $f \in L^1((0, T); X)$  and for some  $\alpha \in (0, 1)$  we have

$$\|f(t) - f(s)\|_X \leq C_{T, \alpha} |t - s|^\alpha, \quad 0 \leq t, s \leq T,$$

then the mild solution (3.2) of (3.1) is in fact a classical solution. In this subsection we shall extend this result to exponentially dichotomous operators generating an analytic bisemigroup.

**Theorem 3.4.** *Let  $S$  be the infinitesimal generator of an analytic bisemigroup on a complex Banach space  $X$  and let  $f \in L^1((-T_2, T_1); X)$ . If  $f$  satisfies the additional condition*

$$\|f(t) - f(s)\|_X \leq \operatorname{const} \cdot |t - s|^\alpha, \quad -T_2 \leq t, s \leq T_1, \quad (3.14)$$

*for some  $\alpha \in (0, 1)$ , then for every  $x_0 \in X$  the mild solution (3.3) of the Cauchy problem (3.4) is classical.*

*Proof.* Consider the vector function  $v$  defined by (3.6). Then

$$\begin{aligned} v(t) &= -S^{-1}[E(0^+; S) - E(t + T_2; S)]f(t) \\ &\quad + S^{-1}[E(0^-; S) - E(t - T_1; S)]f(t) \\ &\quad - \int_{-T_2}^{T_1} E(t - \tau; S)[f(t) - f(\tau)] d\tau. \end{aligned}$$

In the right-hand side the first two terms obviously belong to  $\mathcal{D}(S)$ . Using

$$\|SE(t; S)\| \leq (M/|t|) \quad \text{for } 0 \neq t \in \mathbb{R}$$

as well as (3.14), we easily show  $SE(t - \tau; S)[f(t) - f(\tau)]$  to be Bochner integrable (with respect to the variable  $\tau$ ); hence, by Hille's Lemma 1.2, also the third term belongs to  $\mathcal{D}(S)$ .

Next, for  $-T_2 \leq t < s \leq T_1$  we have by the analyticity of  $E(\cdot; S)$ ,

$$\begin{aligned} \frac{v(s) - v(t)}{s - t} &= \int_0^{t+T_2} \frac{E(s - \tau + T_2; S) - E(t - \tau + T_2; S)}{s - t} f(\tau - T_2) d\tau \\ &\quad + \int_0^{T_1-s} \frac{E(s + \tau - T_1; S) - E(t + \tau - T_1; S)}{s - t} f(T_1 - \tau) d\tau \\ &\quad + \frac{1}{s - t} \int_t^s [E(s - \tau; S) - E(t - \tau; S)] f(\tau) d\tau. \end{aligned}$$

Taking the limit as  $s \rightarrow t^+$ , we obtain

$$\begin{aligned} v'(t) &= \int_0^{t+T_2} SE(t - \tau + T_2; S) f(\tau - T_2) d\tau \\ &\quad + \int_0^{T_1-t} SE(t + \tau - T_1; S) f(T_1 - \tau) d\tau \\ &= \int_{-T_2}^{T_1} SE(t - \tau; S) f(t) d\tau - \int_{-T_2}^{T_1} SE(t - \tau; S) [f(t) - f(\tau)] d\tau \\ &= -[E(0^+; S) - E(t + T_2; S)] f(t) + [E(0^-; S) - E(t - T; S)] f(t) \\ &\quad - \int_{-T_2}^{T_1} SE(t - \tau; S) [f(t) - f(\tau)] d\tau, \end{aligned}$$

where the third term vanishes in the limit as a result of the strong continuity of  $f$ , and the differentiations of the first two terms under the integral signs can be justified by noting that  $\|SE(t; S)\| \leq (M/|t|)$  for  $0 \neq t \in \mathbb{R}$  and employing (3.14). The limit as  $t \rightarrow s^-$  is computed in the same way and yields the same result. Hence,  $v$  is strongly differentiable and the mild solution (3.3) is classical.  $\square$

### 3.2 Characterizing exponential dichotomy

In this section we characterize exponentially dichotomous operators  $S$  in terms of the existence of a unique Bochner integrable solution of the Cauchy problem

$$\begin{cases} u'(t) = Su(t), & 0 \neq t \in \mathbb{R}, \\ u(0^+) - u(0^-) = x_0. \end{cases}$$

This is a direct generalization of the corresponding result characterizing infinitesimal generators of strongly continuous semigroups in terms of weak solutions of the corresponding Cauchy problem [11]. We shall first derive the semigroup result given by Ball [11] and then the analogous result for exponentially dichotomous operators.

We first prove the following characterization of semigroup generators [11].

**Theorem 3.5.** *Let  $A$  be a closed and densely defined linear operator on a complex Banach space  $X$ , and let  $T > 0$ . Suppose that for every  $x_0 \in X$  there exists a unique vector function  $u \in C([0, T]; X)$  such that  $\langle u(\cdot), \phi \rangle$  is absolutely continuous in  $(0, T)$  for each  $\phi \in \mathcal{D}(A^*)$  and*

$$\frac{d}{dt} \langle u(t), \phi \rangle = \langle u(t), A^* \phi \rangle, \quad 0 < t < T \text{ a.e.}, \quad (3.15a)$$

$$u(0^+) = x_0. \quad (3.15b)$$

*Then  $A$  generates a strongly continuous semigroup on  $X$ .*

*Proof.* Using, inductively,  $u(nT)$  as initial data in the Cauchy problem (3.15) (translated to the interval  $(nT, (n+1)T)$ ) we construct a unique weak solution of (3.15) for any  $T > 0$  and not just for the value of  $T$  given in the statement of Theorem 3.5. We may therefore assume, for any  $x_0 \in X$ , the existence of a unique strongly continuous vector function  $u : [0, \infty) \rightarrow X$  such that  $u(0) = x_0$  and, for any  $\phi \in \mathcal{D}(A^*)$ ,  $\langle u(\cdot), \phi \rangle$  is absolutely continuous in  $\mathbb{R}^+$  with a.e. derivative  $\langle u(\cdot), A^* \phi \rangle \in L^1_{\text{loc}}(\mathbb{R}^+)$ .

For  $0 < t_1 < t_2 < +\infty$  and  $\phi \in \mathcal{D}(A^*)$  we obtain by integrating (3.15a)

$$\langle u(t_2), \phi \rangle - \langle u(t_1), \phi \rangle = \int_{t_1}^{t_2} \langle u(\tau), A^* \phi \rangle d\tau = \left\langle \int_{t_1}^{t_2} u(\tau) d\tau, A^* \phi \right\rangle,$$

where we have used that  $u \in C([t_1, t_2]; X)$ . Since in fact  $u \in C([0, T]; X)$  for any  $T > 0$ , we can extend this identity to  $0 \leq t_1 < t_2 < +\infty$ , provided we let  $u(0)$  stand for  $x_0$ . By Proposition 1.1, we then conclude that  $\int_{t_1}^{t_2} u(\tau) d\tau \in \mathcal{D}(A)$  and

$$u(t_2) - u(t_1) = A \int_{t_1}^{t_2} u(\tau) d\tau, \quad 0 \leq t_1 < t_2 < \infty.$$

As a result, for every  $t \in [0, \infty)$  there exists a linear operator  $E(t)$  such that  $E(t)x_0 = u(t)$ . Because  $u : [0, \infty) \rightarrow X$  is continuous for every  $u_0 \in X$ ,  $\{E(t)\}_{t \geq 0}$  is a strongly continuous semigroup on  $X$ .

Let  $B(X \rightarrow X)$  be the infinitesimal generator of this semigroup. Since

$$\frac{d}{dt} \langle E(t)u_0, \phi \rangle = \langle Bu_0, \phi \rangle = \langle u_0, A^* \phi \rangle, \quad t > 0,$$

for any  $u_0 \in \mathcal{D}(B)$ , we have, by Proposition 1.1,  $u_0 \in \mathcal{D}(A)$  and  $Bu_0 = Au_0$  for each  $u_0 \in \mathcal{D}(B)$ . Thus  $\mathcal{D}(B) \subset \mathcal{D}(A)$ . It remains to prove that  $\mathcal{D}(A) \subset \mathcal{D}(B)$ .

Let  $x \in \mathcal{D}(A)$ . Since for each  $t > 0$  and  $\phi \in \mathcal{D}(A^*)$  we have

$$\left\langle \int_0^t E(\tau)x \, d\tau, \phi \right\rangle = \int_0^t \langle E(\tau)x, \phi \rangle \, d\tau = [\langle E(\tau)x, \phi \rangle]_{\tau=0}^t = \langle E(t)x - x, \phi \rangle,$$

we have  $\int_0^t E(\tau)x \, d\tau \in \mathcal{D}(A)$  and

$$A \int_0^t E(\tau)x \, d\tau = E(t)x - x.$$

In the same way we prove that  $\int_0^t E(\tau)Ax \, d\tau \in \mathcal{D}(A)$  and

$$A \int_0^t E(\tau)Ax \, d\tau = E(t)Ax - Ax.$$

Now consider the vector function

$$z(t) = \int_0^t E(\tau)Ax \, d\tau - A \int_0^t E(\tau)x \, d\tau.$$

Then  $z(0) = 0$  and

$$\frac{d}{dt} \langle z(t), \phi \rangle = \langle z(t), A^* \phi \rangle, \quad t > 0,$$

where  $\phi \in \mathcal{D}(A^*)$ . Thus  $z$  is a weak solution of the Cauchy problem (3.15) with zero initial condition and hence vanishes identically. Consequently,

$$\int_0^t E(\tau)Ax \, d\tau = A \int_0^t E(\tau)x \, d\tau, \quad t > 0.$$

Hence

$$\frac{1}{t} [E(t)x - x] = \frac{1}{t} A \int_0^t E(\tau)x \, d\tau = \frac{1}{t} \int_0^t E(\tau)Ax \, d\tau$$

tends to  $Ax$  strongly as  $t \rightarrow 0^+$ . As a result,  $x \in \mathcal{D}(B)$  and  $Bx = Ax$ . We have thus proved that  $\mathcal{D}(A) = \mathcal{D}(B)$  and that  $A$  and  $B$  coincide on their joint domain. We may thus conclude that  $A$  is the infinitesimal generator of the semigroup  $\{E(t)\}_{t \geq 0}$  on  $X$ .  $\square$

We now state and prove the main result of this section.

**Theorem 3.6.** *Let  $S$  be a closed and densely defined linear operator on a complex Banach space  $X$  such that  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = 0\} \subset \rho(S)$ . For each  $x_0 \in X$  let  $u \in [BC(\mathbb{R}^-; X) + BC(\mathbb{R}^+; X)] \cap L^1(\mathbb{R}; X)$  be a unique vector function such that  $\langle u(\cdot), \phi \rangle$  is absolutely continuous in  $\mathbb{R} \setminus \{0\}$  for each  $\phi \in \mathcal{D}(S^*)$  and*

$$\frac{d}{dt} \langle u(t), \phi \rangle = \langle u(t), S^* \phi \rangle, \quad 0 \neq t \in \mathbb{R} \text{ a.e.}, \quad (3.16a)$$

$$u(0^+) - u(0^-) = x_0. \quad (3.16b)$$

*Then  $S$  is exponentially dichotomous.*

*Proof.* Applying the Laplace transform for  $\operatorname{Re} \lambda = 0$ , we obtain from (3.16a),

$$-\langle x_0, \phi \rangle + \lambda \langle \mathcal{L}[u](\lambda), \phi \rangle = \langle \mathcal{L}[u](\lambda), S^* \phi \rangle, \quad \phi \in \mathcal{D}(S^*).$$

Then Proposition 1.1 implies that for  $\operatorname{Re} \lambda = 0$  the vector  $\mathcal{L}[u](\lambda) \in \mathcal{D}(S)$  and

$$(\lambda - S)\mathcal{L}[u](\lambda) = x_0.$$

Thus there exists  $E : \mathbb{R} \times X \rightarrow X$  such that  $E(\cdot, x_0) \in L^1(\mathbb{R}; X)$  for every  $x_0 \in X$  and

$$u(t) = E(t, x_0), \quad 0 \neq t \in \mathbb{R}.$$

Hence, for every  $x_0 \in X$  we have

$$\int_{-\infty}^{\infty} e^{-\lambda \tau} E(\tau, x_0) d\tau = (\lambda - S)^{-1} x_0, \quad \operatorname{Re} \lambda = 0. \quad (3.17)$$

Moreover, for each  $x_0 \in X$  we have

$$E(\cdot, x_0) \in [BC(\mathbb{R}^-; X) \dot{+} BC(\mathbb{R}^+; X)] \cap L^1(\mathbb{R}; X).$$

Now note that for each  $\phi \in \mathcal{D}(S^*)$  and  $\operatorname{Re} \lambda = 0$ ,

$$\begin{aligned} e^{-\lambda t} \langle u(t), \phi \rangle &= - \int_t^{\infty} \langle e^{-\lambda \tau} u(\tau), S^* \phi \rangle d\tau + \lambda \int_t^{\infty} \langle e^{-\lambda \tau} u(\tau), \phi \rangle d\tau \\ &= - \left\langle \int_t^{\infty} e^{-\lambda \tau} u(\tau) d\tau, S^* \phi \right\rangle + \lambda \left\langle \int_t^{\infty} e^{-\lambda \tau} u(\tau) d\tau, \phi \right\rangle \end{aligned}$$

for  $t > 0$  and  $t = 0^+$ , and

$$\begin{aligned} e^{-\lambda t} \langle u(t), \phi \rangle &= \int_{-\infty}^t e^{-\lambda \tau} \langle u(\tau), S^* \phi \rangle d\tau - \lambda \int_{-\infty}^t e^{-\lambda \tau} \langle u(\tau), \phi \rangle d\tau \\ &= \left\langle \int_{-\infty}^t e^{-\lambda \tau} u(\tau) d\tau, S^* \phi \right\rangle - \lambda \left\langle \int_{-\infty}^t e^{-\lambda \tau} u(\tau) d\tau, \phi \right\rangle \end{aligned}$$

for  $t < 0$  and  $t = 0^-$ . Hence, by Proposition 1.1,  $\int_t^{\infty} u(\tau) d\tau \in \mathcal{D}(S)$  and

$$S \int_t^{\infty} e^{-\lambda \tau} u(\tau) d\tau = -e^{-\lambda t} u(t) + \lambda \int_t^{\infty} e^{-\lambda \tau} u(\tau) d\tau, \quad t > 0 \text{ and } t = 0^+,$$

and  $\int_{-\infty}^t u(\tau) d\tau \in \mathcal{D}(S)$  and

$$S \int_{-\infty}^t e^{-\lambda \tau} u(\tau) d\tau = e^{-\lambda t} u(t) + \lambda \int_{-\infty}^t e^{-\lambda \tau} u(\tau) d\tau, \quad t < 0 \text{ and } t = 0^-.$$

Consequently,

$$(\lambda - S)^{-1}u(t) = + \int_t^\infty e^{\lambda(t-\tau)}u(\tau) d\tau, \quad t > 0 \text{ and } t = 0^+, \quad (3.18a)$$

$$(\lambda - S)^{-1}u(t) = - \int_{-\infty}^t e^{\lambda(t-\tau)}u(\tau) d\tau, \quad t < 0 \text{ and } t = 0^-, \quad (3.18b)$$

where  $\operatorname{Re} \lambda = 0$ . Obviously, (3.18a) holds for  $\operatorname{Re} \lambda \geq 0$  and (3.18b) for  $\operatorname{Re} \lambda \leq 0$ .

For each  $x_0 \in X$  we define the (not necessarily closed) linear subspaces

$$\begin{aligned} X_+ &= \{x_0 \in X : E(0^-, x_0) = 0\} = \{x_0 \in X : E(0^+, x_0) = x_0\}, \\ X_- &= \{x_0 \in X : E(0^+, x_0) = 0\} = \{x_0 \in X : E(0^-, x_0) = -x_0\}. \end{aligned}$$

Then uniqueness implies that  $X_+ \cap X_- = \{0\}$ , while the jump condition (3.16b) implies  $X_+ + X_- = X$ . Writing  $P_\pm$  for the (not necessarily bounded) projections of  $X$  onto  $X_\pm$  along  $X_\mp$ , we obtain from (3.18)

$$(\lambda - S)^{-1}P_+x_0 = + \int_0^\infty e^{\lambda(t-\tau)}E(\tau, x_0) d\tau, \quad (3.19a)$$

$$(\lambda - S)^{-1}P_-x_0 = - \int_{-\infty}^0 e^{\lambda(t-\tau)}E(\tau, x_0) d\tau, \quad (3.19b)$$

where  $\operatorname{Re} \lambda = 0$  and  $x_0 \in X$ . Since  $S^{-1}$  is injective, we see from (3.19) that

$$X_+ = \left\{ x_0 \in X : \int_{-\infty}^0 E(\tau, x_0) d\tau = 0 \right\}, \quad (3.20a)$$

$$X_- = \left\{ x_0 \in X : \int_0^\infty E(\tau, x_0) d\tau = 0 \right\}. \quad (3.20b)$$

Let us now prove that  $F : X \rightarrow L^1(\mathbb{R}; X)$  defined by  $Fx_0 = E(\cdot, x_0)$  is bounded. Indeed, let  $\{x_n\}_{n=1}^\infty$  be a sequence in  $X$ , let  $x \in X$ , and let  $g \in L^1(\mathbb{R}; X)$  such that  $\|x_n - x\|_X \rightarrow 0$  and  $\|Fx_n - g\|_{L^1(\mathbb{R}; X)} \rightarrow 0$ . By (3.17) we then have

$$\int_{-\infty}^\infty e^{-\lambda\tau}E(\tau, x_n) d\tau = (\lambda - S)^{-1}x_n, \quad \operatorname{Re} \lambda = 0.$$

Taking the limit in the norm of  $X$  we get

$$\int_{-\infty}^\infty e^{-\lambda t}g(t) dt = (\lambda - S)^{-1}x, \quad \operatorname{Re} \lambda = 0.$$

Applying an arbitrary functional  $\phi \in X^*$  we obtain

$$\int_{-\infty}^\infty e^{-\lambda t}\langle g(t), \phi \rangle dt = \langle (\lambda - S)^{-1}x, \phi \rangle, \quad \operatorname{Re} \lambda = 0.$$

By the uniqueness of the Laplace transform [164], we have

$$\langle g(t), \phi \rangle = \langle E(t, x), \phi \rangle$$

for a.e.  $t \in \mathbb{R}$  and each  $g \in L^1(\mathbb{R}; X)$ . Thus  $g(t) = E(t, x)$  for a.e.  $t \in \mathbb{R}$ . By the Closed Graph Theorem we now conclude that  $F$  is bounded.

Because of (3.20), the boundedness of  $F$  implies that  $X_+$  and  $X_-$  are closed linear subspaces of  $X$ , which implies the boundedness of the projections  $P_{\pm}$ . Further, (3.16) implies that

$$\mathcal{D}(S) = [\mathcal{D}(S) \cap X_+] \dot{+} [\mathcal{D}(S) \cap X_-].$$

We can therefore define  $S_{\pm}(X_{\pm} \rightarrow X_{\pm})$  as the restrictions of  $S$  to  $X_{\pm}$  with dense domains  $\mathcal{D}(S) \cap X_{\pm}$ , resulting in closed and densely defined linear operators  $S_{\pm}$  on  $X_{\pm}$ . Then for any  $u_0 \in X_{\pm}$  the Cauchy problems

$$\begin{cases} u'(t) = \pm S_{\pm} u(t), & t > 0, \\ u(0^+) = u_0, \end{cases}$$

have a unique weak solution in the following sense:

- 1)  $u : [0, \infty) \rightarrow X_{\pm}$  is strongly continuous,
- 2) for any  $\phi \in \mathcal{D}(S_{\pm}^*)$  the function  $\langle u(\cdot), \phi \rangle$  is absolutely continuous in  $\mathbb{R}^+$  and has as its a.e. derivative  $\pm \langle u(\cdot), S_{\pm}^* \phi \rangle$ , and
- 3)  $u(0^+) = u_0$ .

As a result of Theorem 3.5,  $\pm S_{\pm}$  generates a strongly continuous semigroup on  $X_{\pm}$ . Moreover, we have

$$\int_0^{\infty} \|e^{\pm t S_{\pm}} u_0\|_{X_{\pm}} dt < \infty,$$

irrespective of the choice of  $u_0 \in X_{\pm}$ . According to Theorem 1.4, either semigroup is exponentially decreasing. Putting

$$E(t; S)x_0 = \begin{cases} e^{t S_+} P_+ x_0, & t > 0, \\ -e^{t S_-} P_- x_0, & t < 0, \end{cases}$$

we see that  $E(t, S)x_0 = E(t, x_0)$  for any  $0 \neq t \in \mathbb{R}$ , while there exists  $\varepsilon > 0$  such that for each  $x_0 \in X$  and  $0 \neq t \in \mathbb{R}$ ,

$$\|E(t, S)x_0\| \leq \text{const.} e^{-\varepsilon|t|} \|x_0\|.$$

Consequently, as a result of Theorem 1.7 we conclude that  $S$  is exponentially dichotomous, as claimed.  $\square$



## Chapter 4

# Riccati Equations and Wiener-Hopf Factorization

In this chapter we connect bounded additive exponentially dichotomous perturbations  $S$  of an exponentially dichotomous operator  $S_0$  to left and right canonical Wiener-Hopf factorizations of the fractional linear function

$$W(\lambda) = (\lambda - S_0)^{-1}(\lambda - S).$$

In fact, we prove the so-called triple equivalence of (i) canonical factorizability, (ii) a decomposition of the underlying Banach space  $X$  of the type

$$\operatorname{Im} E(0^\pm; S_0) + \operatorname{Im} E(0^\mp; S) = X,$$

and (iii) the unique solvability of a vector-valued Wiener-Hopf equation with convolution kernel  $E(\cdot; S_0)\Gamma$ , where  $\Gamma = S - S_0$ . In particular, if  $S_0$  and  $S$  are written in block matrix form with respect to the decomposition induced by the separating projection of  $S_0$  and the bounded additive perturbation  $\Gamma$  is off-diagonal with respect to this decomposition, we convert the equivalent statements derived into an existence result for certain solutions of Riccati equations in  $\mathcal{L}(X)$ . We conclude this chapter with perturbation results on the solutions of these Riccati equations.

### 4.1 Canonical factorization and perturbation

In this section we define left and right (quasi-)canonical factorizations of operator functions on the imaginary line and prove the equivalence between the existence of a (quasi-)canonical factorization, a certain decomposition of the underlying Banach space, and the unique solvability of a vector-valued Wiener-Hopf equation. This will be done both for Bochner and Pettis integrable convolution kernels, the

latter only in a Hilbert space setting. We pay particular attention to operator functions close to the identity and operator functions with positive definite real part.

The literature abounds with mathematical models in which the so-called triple equivalence between

- (i) canonical factorizability,
- (ii) direct sum decomposition, and
- (iii) unique solvability of a vector-valued Wiener-Hopf equation,

comes to the fore. This triple equivalence is already implicit in the various theories of characteristic operator functions and operator models ([36, 37, 150] and references therein), although the selfadjointness or unitarity of the operator functions involved tends to obscure the basic principle. In linear systems theory the triple equivalence models cascade decomposition of a continuous time noncausal linear system into a causal and an anticausal system, irrespective of whether we deal with finite-dimensional systems [15, 16, 95, 18], infinite-dimensional systems with bounded input and output operators [17], or Pritchard-Salamon systems [97]. Commonly, in linear systems theory the time evolution of the linear system in state space is governed by a hyperbolic semigroup, where the absence of imaginary eigenvalues of the infinitesimal generator leads to the triple equivalence [52, 148, 143, 13, 51]. Another area involving triple equivalence is the study of abstract kinetic equations [152, 77]. In this section we link triple equivalence more explicitly to exponential dichotomy in arbitrary complex Banach spaces [134].

#### 4.1.1 Canonical factorization

Let  $Y$  be a complex Banach space. Suppose  $W$  is an operator function defined on the extended imaginary line with values in  $\mathcal{L}(Y)$ , which is continuous in the norm on  $i\mathbb{R}$  and strongly continuous at  $\pm i\infty$ . Then

$$W(\lambda) = W_l(\lambda)W_r(\lambda), \quad \lambda \in i\mathbb{R} \cup \{i\infty\}, \quad (4.1)$$

is called a *left quasi-canonical factorization* of  $W$  with respect to the imaginary line if:

1.  $W_l$  and  $W_r$  extend to operator functions that are continuous in the norm on the left and right closed half-planes (excluding  $\infty$ ), analytic on the left and right open half-planes, and strongly continuous on the left and right closed half-planes (including  $\infty$ ), respectively.
2.  $W_l(\lambda)$  and  $W_r(\lambda)$  have bounded inverses for all  $\lambda$  in the closed left and right half-planes (including  $\infty$ ), respectively.
3.  $W_l(\cdot)^{-1}$  and  $W_r(\cdot)^{-1}$  are strongly continuous on the left and right closed half-planes (including  $\infty$ ), respectively.

A factorization of  $W$  of the form

$$W(\lambda) = W_r(\lambda)W_l(\lambda), \quad \lambda \in i\mathbb{R} \cup \{i\infty\}, \quad (4.2)$$

where the factors  $W_l$  and  $W_r$  have the properties 1–3 stated above, is called a *right quasi-canonical factorization* of  $W$  with respect to the imaginary line. If  $W$  is assumed continuous in the norm on the extended imaginary line and the continuity conditions in 1–3 hold with respect to the norm topology instead of the strong operator topology (thus making obsolete condition 3), the above factorizations are called left and right *canonical*.

We failed to find the following useful result in the literature. It is not known if it holds in a general Banach space setting. We restrict ourselves to left canonical factorization, though the analogous result for right canonical factorization is equally true.

**Proposition 4.1.** *Let  $X$  be a complex Hilbert space, and suppose the operator function defined by*

$$W(\lambda) = I_X + \int_{-\infty}^{\infty} e^{-\lambda t} F(t) dt, \quad \lambda \in i\mathbb{R} \cup \{\infty\},$$

*with  $F \in L^1(\mathbb{R}; \mathcal{L}(X))$ , has a left canonical factorization of the form (4.1). Then there exist an invertible operator  $G \in \mathcal{L}(X)$  and operator functions  $\delta_l, \delta_r, \gamma_l, \gamma_r \in L^1(\mathbb{R}^+; \mathcal{L}(X))$  such that*

$$\begin{aligned} W_r(\lambda) &= G \left[ I_X + \int_0^{\infty} e^{-\lambda t} \delta_r(t) dt \right], \\ W_l(\lambda) &= \left[ I_X + \int_{-\infty}^0 e^{-\lambda t} \delta_l(-t) dt \right] G^{-1}, \\ W_r(\lambda)^{-1} &= \left[ I_X + \int_0^{\infty} e^{-\lambda t} \gamma_r(t) dt \right] G^{-1}, \\ W_l(\lambda)^{-1} &= G \left[ I_X + \int_{-\infty}^0 e^{-\lambda t} \gamma_l(-t) dt \right]. \end{aligned}$$

*Proof.* With no loss of generality we assume that  $W_l(\pm\infty) = I_X$  and  $W_r(\pm i\infty) = I_X$ . For every  $\delta > 0$  there exist  $\tilde{W}_l$  and  $\tilde{W}_r$  such that

$$\begin{aligned} \tilde{W}_r(\lambda)^{-1} &= I_X + \int_0^{\infty} e^{-\lambda t} \tilde{\gamma}_r(t) dt, \\ \tilde{W}_l(\lambda)^{-1} &= I_X + \int_{-\infty}^0 e^{-\lambda t} \tilde{\gamma}_l(-t) dt, \end{aligned}$$

for certain  $\tilde{\gamma}_l, \tilde{\gamma}_r \in L^1(\mathbb{R}^+; \mathcal{L}(X))$  and

$$\sup_{\lambda \in \mathbb{R}} \left( \|W_l(\lambda)^{-1} - \tilde{W}_l(\lambda)^{-1}\| + \|W_l(\lambda)^{-1} - \tilde{W}_l(\lambda)^{-1}\| \right) < \delta.$$

Put  $Z(\lambda) = \tilde{W}_l(\lambda)^{-1}W(\lambda)\tilde{W}_r(\lambda)^{-1}$ . Then there exists  $z \in L^1(\mathbb{R}; \mathcal{L}(X))$  such that

$$Z(\lambda) = I_X + \int_{-\infty}^{\infty} e^{-\lambda t} z(t) dt,$$

while

$$\sup_{\lambda \in i\mathbb{R}} \|Z(\lambda) - I_X\| < \delta M(M + \delta) + M^2 \delta,$$

where  $M$  exceeds  $\|W(\lambda)\|$ ,  $\|W_l(\lambda)^{-1}\|$ , and  $\|W_r(\lambda)^{-1}\|$  for any  $\lambda \in i\mathbb{R}$ . Choosing  $0 < \delta < -M + (M^2 + M^{-1})^{1/2}$ , we obtain

$$\sup_{\lambda \in i\mathbb{R}} \|Z(\lambda) - I_X\| < 1.$$

A well-known result by Gohberg and Leiterer [75] on the canonical factorizability of operator functions on the circle or line differing from the identity by less than 1 in the supremum norm (applicable because  $X$  is a complex Hilbert space) then implies the existence of  $z_r, z_l, w_r, w_l \in L^1(\mathbb{R}^+; \mathcal{L}(X))$  such that

$$\begin{aligned} Z_r(\lambda) &= I_X + \int_0^{\infty} e^{-\lambda t} z_r(t) dt, \\ Z_l(\lambda) &= I_X + \int_{-\infty}^0 e^{-\lambda t} z_l(-t) dt, \\ Z_r(\lambda)^{-1} &= I_X + \int_0^{\infty} e^{-\lambda t} w_r(t) dt, \\ Z_l(\lambda)^{-1} &= I_X + \int_{-\infty}^0 e^{-\lambda t} w_l(-t) dt, \end{aligned}$$

while

$$Z(\lambda) = Z_l(\lambda)Z_r(\lambda), \quad \lambda \in i\mathbb{R} \cup \{\infty\}.$$

Then  $W_l = \tilde{W}_l Z_l$  and  $W_r = Z_r \tilde{W}_r$  lead to a factorization as described in the statement of this proposition.  $\square$

### 4.1.2 When perturbations lead to Bochner integrable kernels

Suppose that  $E(\cdot; S_0)\Gamma \in L^1(\mathbb{R}; \mathcal{L}(X))$ , which is the case if (1)  $\Gamma$  is a compact operator (cf. Subsection 2.2.1), or if (2)  $E(\cdot; S_0)$  is analytic (cf. Subsection 2.2.2), or if (3)  $E(\cdot; S_0)$  is immediately norm continuous (cf. Subsection 2.2.3). In these cases the operator of convolution by  $E(\cdot; S_0)\Gamma$  from the left, i.e.,

$$(T\psi)(t) = \int_{-\infty}^{\infty} E(t-s; S_0)\Gamma\psi(s) ds, \quad (4.3)$$

is bounded on any of the Banach function spaces

$$L^p(\mathbb{R}; X) \ (1 \leq p < \infty), \quad BC(\mathbb{R}; X), \quad \text{and} \quad BC(\mathbb{R}^-; X) + BC(\mathbb{R}^+; X).$$

We call these Banach function spaces *allowable* and denote them by  $E(\mathbb{R}; X)$ . The term “allowable” is also used for the spaces  $L^p(\mathbb{R}^\pm; X)$  ( $1 \leq p < \infty$ ) and  $BC(\mathbb{R}^\pm; X)$ .

We first need the following crucial lemma.

**Lemma 4.2.** *Let  $S_0$  be exponentially dichotomous on the complex Banach space  $X$ . Then the operator  $L$  defined by*

$$(L\psi)(t) = \int_{-\infty}^{\infty} E(t-s; S_0)\psi(s) ds$$

*is bounded on the Banach function spaces  $L^p(\mathbb{R}; X)$  ( $1 \leq p < \infty$ ),  $BC(\mathbb{R}; X)$ , and  $BC(\mathbb{R}^-; X) \dot{+} BC(\mathbb{R}^+; X)$ .*

Observe that Lemma 4.2 on  $L^1(\mathbb{R}; X)$  is immediate from Lemma 2.10.

*Proof.* For each  $t \in \mathbb{R}$  the function  $s \mapsto E(t-s; S_0)\chi(s)$  is strongly measurable if  $\chi$  is an  $X$ -valued simple function. In this case we easily show that  $L\chi \in L^p(\mathbb{R}; X)$  ( $1 \leq p \leq \infty$ ) and

$$\|L\chi\|_{L^p(\mathbb{R}; X)} \leq C\|\chi\|_{L^p(\mathbb{R}; X)}, \quad (4.4)$$

where  $C = \text{const.} \int_{-\infty}^{\infty} e^{-r|t|} dt$ . Here  $\|E(t; S_0)\| \leq \text{const.} e^{-r|t|}$  for  $0 \neq t \in \mathbb{R}$ . Using the density of the simple  $X$ -valued functions, we obtain Lemma 4.2 for  $L^p(\mathbb{R}; X)$  ( $1 \leq p < \infty$ ), but not for  $L^\infty(\mathbb{R}; X)$ .

Now observe that the integral in (4.4) is a Bochner integral if the vector function  $\chi \in BC(\mathbb{R}^-; X) \dot{+} BC(\mathbb{R}^+; X)$ . Using the Dominated Convergence Theorem we then easily prove that  $L\chi \in BC(\mathbb{R}^-; X) \dot{+} BC(\mathbb{R}^+; X)$ .  $\square$

The following two results have been proved in their present form in [134].

**Theorem 4.3.** *Let  $S_0$  be an exponentially dichotomous operator on a complex Banach space  $X$ , let  $\Gamma \in \mathcal{L}(X)$ , and let  $E(\cdot; S_0)\Gamma \in L^1(\mathbb{R}; \mathcal{L}(X))$ . Suppose  $S = S_0 + \Gamma$  satisfies  $\{\lambda \in \mathbb{C} : |\text{Re } \lambda| \leq \varepsilon\} \subset \rho(S)$  for some  $\varepsilon > 0$ . Then the following statements are equivalent:*

(a) *The operator function*

$$W(\lambda) = (\lambda - S_0)^{-1}(\lambda - S) = I_X - (\lambda - S_0)^{-1}\Gamma, \quad |\text{Re } \lambda| \leq \varepsilon, \quad (4.5)$$

*has a left canonical factorization with respect to the imaginary axis of the form (4.1), where for certain  $\gamma_l, \gamma_r \in L^1(\mathbb{R}^+; \mathcal{L}(X))$ ,*

$$W_r(\lambda)^{-1} = I_X + \int_0^\infty e^{-\lambda t} \gamma_r(t) dt, \quad (4.6a)$$

$$W_l(\lambda)^{-1} = I_X + \int_{-\infty}^0 e^{-\lambda t} \gamma_l(-t) dt. \quad (4.6b)$$

(b) *We have the decomposition*

$$\operatorname{Im} E(0^+; S) \dot{+} \operatorname{Im} E(0^-; S_0) = X. \quad (4.7)$$

(c) *For some (and hence every) allowable  $E(\mathbb{R}^+; X)$ , the vector-valued Wiener-Hopf equation*

$$\phi(t) - \int_0^\infty E(t-s; S_0) \Gamma \phi(s) ds = g(t), \quad t \in \mathbb{R}^+, \quad (4.8)$$

*is uniquely solvable in  $E(\mathbb{R}^+; X)$  for any  $g \in E(\mathbb{R}^+; X)$ .*

(d) *For some (and hence every) allowable  $E(\mathbb{R}^+; X)$ , the vector-valued Wiener-Hopf equation*

$$\psi(t) - \int_0^\infty \Gamma E(t-s; S_0) \psi(s) ds = h(t), \quad t \in \mathbb{R}^+, \quad (4.9)$$

*is uniquely solvable in  $E(\mathbb{R}^+; X)$  for any  $h \in E(\mathbb{R}^+; X)$ .*

(e) *Consider  $\Gamma_1 \in \mathcal{L}(X_0, X)$  and  $\Gamma_2 \in \mathcal{L}(X, X_0)$  such that  $\Gamma = \Gamma_1 \Gamma_2$ . Then for some (and hence every) allowable  $E(\mathbb{R}^+; X)$ , the vector-valued Wiener-Hopf equation*

$$\varphi(t) - \int_0^\infty \Gamma_2 E(t-s; S_0) \Gamma_1 \varphi(s) ds = f(t), \quad t \in \mathbb{R}^+, \quad (4.10)$$

*is uniquely solvable in  $E(\mathbb{R}^+; X_0)$  for any  $f \in E(\mathbb{R}^+; X_0)$ .*

According to Proposition 4.1, the conditions (4.6) on the inverses of the factors in the left canonical factorization are redundant if  $X$  is a complex Hilbert space.

*Proof.* Note that under the above hypotheses  $S$  is exponentially dichotomous.

(c)  $\iff$  (d)  $\iff$  (e) It follows immediately from Lemma 4.2 that the operator  $L_+$  defined by

$$(L_+ \psi)(t) = \int_0^\infty E(t-s; S_0) \psi(s) ds$$

is bounded on all of the allowable Banach function spaces  $E(\mathbb{R}^+; X)$ . Moreover, (4.8)-(4.10) can be written in the concise form

$$\begin{aligned} \phi - L_+ \Gamma \phi &= g, \\ \psi - \Gamma L_+ \psi &= h, \\ \varphi - \Gamma_2 L_+ \Gamma_1 \varphi &= f. \end{aligned}$$

It is then immediate that all three equations are uniquely solvable on the analogous allowable Banach function space if at least one is.

**(b)  $\implies$  (a)** Suppose (4.7) is true and let  $\Pi$  denote the projection of  $X$  onto  $\text{Im } E(0^+; S)$  along  $\text{Im } E(0^-; S_0)$ . Then standard methods [15, 19, 95, 18] or mere inspection imply that

$$W(\lambda) = [I_X - (\lambda - S_0)^{-1}(I - \Pi)\Gamma] [I_X - \Pi(\lambda - S_0)^{-1}\Gamma], \quad (4.11)$$

where

$$[I_X - (\lambda - S_0)^{-1}(I - \Pi)\Gamma]^{-1} = I_X + (I - \Pi)(\lambda - S)^{-1}\Gamma, \quad (4.12a)$$

$$[I_X - \Pi(\lambda - S_0)^{-1}\Gamma]^{-1} = I_X + (\lambda - S)^{-1}\Pi\Gamma, \quad (4.12b)$$

is a left canonical factorization of  $W$  with respect to the imaginary line.

**(a)  $\implies$  (c)** Suppose the operator function  $W$  defined by (4.5) has a left canonical factorization  $W = W_l W_r$  with respect to the imaginary axis, where there exist  $\gamma_l, \gamma_r \in L^1(\mathbb{R}^+; \mathcal{L}(X))$  such that (4.6) hold. Then standard methods (cf. [72], [68, Sec. I.8], [69, Ch. XIII]) show that

$$\phi(t) = g(t) + \int_0^\infty \gamma(t, s)g(s) ds,$$

where

$$\gamma(t, s) = \begin{cases} \gamma_r(t - s) + \int_0^s \gamma_r(t - \tau)\gamma_l(s - \tau) d\tau, & 0 \leq s < t < \infty, \\ \gamma_l(s - t) + \int_0^t \gamma_r(t - \tau)\gamma_l(s - \tau) d\tau, & 0 \leq t < s < \infty, \end{cases}$$

is the unique solution of (4.8) in  $L^p(\mathbb{R}^+; X)$  for each  $g \in L^p(\mathbb{R}^+; X)$  ( $1 \leq p \leq \infty$ ). This solution belongs to  $BC(\mathbb{R}^+; X)$  whenever  $g \in BC(\mathbb{R}^+; X)$ .

**(c)  $\implies$  (b)** Suppose (4.8) has a unique solution  $\phi \in BC(\mathbb{R}^+; X)$  for each  $g \in BC(\mathbb{R}^+; X)$ . Let  $\phi(\cdot, x)$  be this solution if  $g(t) = E(t; S_0)x$ , given  $x \in X$ . For  $t > 0$  and  $u \geq 0$  we now compute

$$\begin{aligned} \phi(t + u, x) &= \int_0^\infty E(t - s; S_0)\Gamma\phi(s + u, x) ds \\ &= \phi(t + u, x) - \int_u^\infty E(t + u - s; S_0)\Gamma\phi(s, x) ds \\ &= E(t + u; S_0)x + \int_0^u E(t + u - s; S_0)\Gamma\phi(s, x) ds \\ &= E(t; S_0) \left[ E(u; S_0)x + \int_0^u E(u - s; S_0)\Gamma\phi(s, x) ds \right] \\ &= E(t; S_0) \left[ E(u; S_0)x + \int_0^\infty E(u - s; S_0)\Gamma\phi(s, x) ds \right] \\ &= E(t; S_0)\phi(u, x), \end{aligned}$$

where the penultimate transition follows from  $E(t; S_0)E(u; S_0) = 0$  whenever  $tu < 0$ . Hence, we have derived the product rule

$$\phi(t + u, x) = \phi(t, \phi(u, x)), \quad t, u \in \mathbb{R}^+, x \in X.$$

Letting  $\Pi$  stand for the linear operator defined by  $\Pi x = \phi(0^+, x)$  for  $x \in X$ , we find from the product rule that  $\Pi$  is a bounded projection on  $X$  whose kernel coincides with  $\text{Im } E(0^-; S_0)$ .

If  $y \in \mathcal{D}(S) = \mathcal{D}(S_0)$  (so that  $E(t; S)y \in \mathcal{D}(S)$ ), we compute for  $t > 0$ ,

$$\begin{aligned} E(t; S)y &- \int_0^\infty E(t-s; S_0)\Gamma E(s; S)y \, ds \\ &= E(t; S)y - \left( \int_0^t + \int_t^\infty \right) \frac{\partial}{\partial s} \{E(t-s; S_0)E(s; S)y\} \, ds \\ &= E(t; S)y - E(0^+; S_0)E(t; S)y + E(0^-; S_0)E(t; S)y + E(t; S_0)E(0^+; S)y \\ &= E(t; S_0)E(0^+; S)y. \end{aligned}$$

Hence for all  $y \in \mathcal{D}(S_0) = \mathcal{D}(S)$  we have

$$\phi(t, E(0^+; S)y) = E(t; S)y.$$

By continuous extension we then get  $\text{Im } E(0^+; S) \subset \text{Im } \Pi$ .

It remains to prove that  $\text{Im } \Pi \subset \text{Im } E(0^+; S)$ . For  $z \in \mathcal{D}(S^*)$  ( $\subset X^*$ , the dual space of  $X$ ), we easily get

$$\frac{d}{dt} \langle \phi(t, x), z \rangle = \langle \phi(t, x), S_0^* z \rangle + \langle \Gamma \phi(t, x), z \rangle = \langle \phi(t, x), S^* z \rangle.$$

Consequently,

$$\begin{aligned} 0 &= \int_0^\infty e^{-\lambda t} \left\{ \frac{d}{dt} \langle \phi(t, x), z \rangle - \langle \phi(t, x), S^* z \rangle \right\} dt \\ &= [e^{-\lambda t} \langle \phi(t, x), z \rangle]_{t=0}^\infty + \int_0^\infty e^{-\lambda t} \langle \phi(t, x), (\lambda - S^*)z \rangle dt \\ &= -\langle \Pi x, z \rangle + \langle \hat{\phi}(\lambda, x), (\lambda - S^*)z \rangle, \end{aligned}$$

where  $\hat{\phi}(\lambda, x)$  is the Laplace transform of  $\phi(\cdot, x)$ . Since  $S$  is closed and densely defined, by Proposition 1.1 it follows that  $\hat{\phi}(\lambda, x) \in \mathcal{D}(S)$  and

$$\Pi x = (\lambda - S)\hat{\phi}(\lambda, x),$$

whence

$$\hat{\phi}(\lambda, x) = (\lambda - S)^{-1} \Pi x, \quad \text{Re } \lambda < 0.$$

Because  $S$  is exponentially dichotomous, the last equality implies that  $\Pi x \in \text{Im } E(0^+; S)$ , as claimed.  $\square$



The following theorem can be proved analogously.

**Theorem 4.4.** *Let  $S_0$  be an exponentially dichotomous operator on a complex Banach space  $X$ , let  $\Gamma \in \mathcal{L}(X)$ , and let  $E(\cdot; S_0)\Gamma \in L^1(\mathbb{R}; \mathcal{L}(X))$ . Suppose  $S = S_0 + \Gamma$  satisfies  $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq \varepsilon\} \subset \rho(S)$  for some  $\varepsilon > 0$ . Then the following statements are equivalent:*

(a) *The operator function*

$$W(\lambda) = (\lambda - S_0)^{-1}(\lambda - S) = I_X - (\lambda - S_0)^{-1}\Gamma, \quad |\operatorname{Re} \lambda| \leq \varepsilon, \quad (4.13)$$

*has a right canonical factorization with respect to the imaginary axis of the form (4.2), where (4.6) is valid for certain  $\gamma_l, \gamma_r \in L^1(\mathbb{R}^+; \mathcal{L}(X))$ .*

(b) *We have the decomposition*

$$\operatorname{Im} E(0^-; S) + \operatorname{Im} E(0^+; S_0) = X. \quad (4.14)$$

(c) *For some (and hence every) allowable  $E(\mathbb{R}^-; X)$ , the vector-valued Wiener-Hopf equation*

$$\phi(t) - \int_{-\infty}^0 E(t-s; S_0)\Gamma\phi(s) ds = g(t), \quad t \in \mathbb{R}^-, \quad (4.15)$$

*is uniquely solvable in  $E(\mathbb{R}^-; X)$  for any  $g \in E(\mathbb{R}^-; X)$ .*

(d) *For some (and hence every) allowable  $E(\mathbb{R}^-; X)$ , the vector-valued Wiener-Hopf equation*

$$\psi(t) - \int_{-\infty}^0 \Gamma E(t-s; S_0)\psi(s) ds = h(t), \quad t \in \mathbb{R}^-, \quad (4.16)$$

*is uniquely solvable in  $E(\mathbb{R}^-; X)$  for any  $h \in E(\mathbb{R}^-; X)$ .*

(e) *Consider  $\Gamma_1 \in \mathcal{L}(X_0, X)$  and  $\Gamma_2 \in \mathcal{L}(X, X_0)$  such that  $\Gamma = \Gamma_1\Gamma_2$ . Then for some (and hence every) allowable  $E(\mathbb{R}^-; X)$ , the vector-valued Wiener-Hopf equation*

$$\varphi(t) - \int_{-\infty}^0 \Gamma_2 E(t-s; S_0)\Gamma_1\varphi(s) ds = f(t), \quad t \in \mathbb{R}^-, \quad (4.17)$$

*is uniquely solvable in  $E(\mathbb{R}^-; X_0)$  for any  $f \in E(\mathbb{R}^-; X_0)$ .*

**Corollary 4.5.** *Let  $S_0$  be an exponentially dichotomous operator on a complex Hilbert space  $X$ , let  $\Gamma \in \mathcal{L}(X)$ , and let  $E(\cdot; S_0)\Gamma \in L^1(\mathbb{R}; \mathcal{L}(X))$ . Suppose  $S = S_0 + \Gamma$  satisfies  $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq \varepsilon\} \subset \rho(S)$  for some  $\varepsilon > 0$ . Suppose that for some complex Hilbert space  $X_0$  and certain  $\Gamma_1 \in \mathcal{L}(X, X_0)$  and  $\Gamma_2 \in \mathcal{L}(X_0, X)$  with  $\Gamma = \Gamma_1\Gamma_2$  one of the following statements is true:*

1. We have

$$\sup_{\operatorname{Re} \lambda=0} \|\Gamma_2(\lambda - S_0)^{-1}\Gamma_1\|_{\mathcal{L}(X)} < 1. \quad (4.18)$$

2. There exists  $\delta > 0$  such that

$$\operatorname{Re} \langle [I_{X_0} + \Gamma_2(\lambda - S_0)^{-1}\Gamma_1] x, x \rangle \geq \delta \|x\|^2 \quad (4.19)$$

for each  $x \in X_0$ .

Then every single one of the following statements is true:

- (a) The operator function  $W$  in (4.5) has a left and a right canonical factorization with respect to the imaginary axis.
- (b) We have both of the decompositions (4.7) and (4.14).
- (c) For some (and hence every) allowable function space  $E(\mathbb{R}^\pm; X)$ , the vector-valued Wiener-Hopf equation (4.8) ((4.15), respectively) has a unique solution in  $E(\mathbb{R}^\pm; X)$  for any  $g \in E(\mathbb{R}^\pm; X)$ .
- (d) For some (and hence every) allowable function space  $E(\mathbb{R}^\pm; X)$ , the vector-valued Wiener-Hopf equation (4.9) ((4.16), respectively) has a unique solution in  $E(\mathbb{R}^\pm; X)$  for any  $h \in E(\mathbb{R}^\pm; X)$ .
- (e) For some (and hence every) allowable function space  $E(\mathbb{R}^\pm; X_0)$ , the vector-valued Wiener-Hopf equation (4.10) ((4.17), respectively) has a unique solution in  $E(\mathbb{R}^\pm; X_0)$  for any  $f \in E(\mathbb{R}^\pm; X_0)$ .

Because of Proposition 4.1 we can state condition (a) in the present rather elementary form.

*Proof.* It suffices to prove part (e) for  $p = 2$ . Because of the unitarity of the Fourier transform it is easy to see that on  $L^2(\mathbb{R}^\pm; X_0)$  the norms of the convolution operators in (4.10) and (4.17) are bounded above by the left-hand side of (4.18) [cf. Lemma 2.11], which implies our result. We could as well have assumed (4.19), since this hypothesis is equivalent to the existence of a constant  $c > 0$  for which

$$\|c(I_{X_0} + \Gamma_2(\lambda - S_0)^{-1}\Gamma_1) - I_{X_0}\| < 1.$$

Then  $cW$  has a left and right canonical factorization with respect to the imaginary line and so does  $W$ .  $\square$

### 4.1.3 When perturbations lead to Pettis integrable kernels

If  $S_0$  is exponentially dichotomous,  $\Gamma$  is bounded on a complex Banach space  $X$ , and no additional assumptions are made, the convolution kernel  $E(\cdot; S_0)\Gamma$  is generally only Pettis integrable. In this case the convolution operator  $T$  defined by (4.3) is bounded on  $L^2(\mathbb{R}; X)$ , provided  $X$  is a complex Hilbert space (cf. Lemma 2.11). In this subsection we therefore assume  $X$  to be a complex Hilbert space and

let  $L^2(\mathbb{R}; X)$  be the only “allowable” Banach function space. In this case, since the operator  $L$  given in the statement of Lemma 4.2 can be represented by the diagram

$$L^2(\mathbb{R}; X) \xrightarrow{\mathcal{F}} L^2(\mathbb{R}; X) \xrightarrow{(\lambda - S_0)^{-1}} L^2(\mathbb{R}; X) \xrightarrow{\mathcal{F}^{-1}} L^2(\mathbb{R}; X),$$

its boundedness on  $L^2(\mathbb{R}; X)$  follows from the unitarity of the Fourier transform. Moreover, if  $S = S_0 + \Gamma$  satisfies  $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq \varepsilon\} \subset \rho(S)$  for some  $\varepsilon > 0$ , then  $S$  is exponentially dichotomous, as a result of Theorem 2.13.

We now modify Theorem 4.3 to the present framework. We omit the obvious statements of the analogs of Theorem 4.4 and Corollary 4.5.

**Theorem 4.6.** *Let  $S_0$  be an exponentially dichotomous operator on a complex Hilbert space  $X$ , let  $\Gamma \in \mathcal{L}(X)$ , and let  $S = S_0 + \Gamma$  satisfy  $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq \varepsilon\} \subset \rho(S)$  for some  $\varepsilon > 0$ . Then the following statements are equivalent:*

- (a) *The operator function defined by (4.5) has a left quasi-canonical factorization with respect to the imaginary axis of the form (4.1), where for certain  $\gamma_l, \gamma_r : \mathbb{R}^+ \times X \rightarrow X$  such that  $\gamma_l(\cdot, x), \gamma_r(\cdot, x) \in L^1(\mathbb{R}^+; X)$  for each  $x \in X$ , we have for  $x \in X$ ,*

$$W_r(\lambda)^{-1}x = x + \int_0^\infty e^{-\lambda t} \gamma_r(t, x) dt, \quad (4.20a)$$

$$W_l(\lambda)^{-1}x = x + \int_{-\infty}^0 e^{-\lambda t} \gamma_l(-t, x) dt. \quad (4.20b)$$

- (b) *We have the decomposition (4.7).*
- (c) *For any  $g \in L^2(\mathbb{R}^+; X)$  the vector-valued Wiener-Hopf equation (4.8) is uniquely solvable in  $L^2(\mathbb{R}^+; X)$ .*
- (d) *For any  $h \in L^2(\mathbb{R}^+; X)$  the vector-valued Wiener-Hopf equation (4.9) is uniquely solvable in  $L^2(\mathbb{R}^+; X)$ .*
- (e) *Let  $X_0$  be a complex Hilbert space,  $\Gamma_1 \in \mathcal{L}(X_0, X)$ , and  $\Gamma_2 \in \mathcal{L}(X, X_0)$  such that  $\Gamma = \Gamma_1 \Gamma_2$ . Then for any  $f \in L^2(\mathbb{R}^+; X_0)$  the vector-valued Wiener-Hopf equation (4.10) is uniquely solvable in  $L^2(\mathbb{R}^+; X_0)$ .*

*Proof.* The equivalence of (c), (d), and (e) as well as the implication **(b)**  $\implies$  **(a)** can be established as in the proof of Theorem 4.3.

**(a)**  $\implies$  **(c)** Suppose the operator function  $W$  defined by (4.5) has a left quasi-canonical factorization  $W = W_l W_r$  with respect to the imaginary axis, where there exist  $\gamma_l, \gamma_r : \mathbb{R}^+ \times X \rightarrow X$  such that for each  $x \in X$  the vector functions  $\gamma_l(\cdot, x)$  and  $\gamma_r(\cdot, x)$  belong to  $L^1(\mathbb{R}^+; X)$  and (4.20) are valid for every  $x \in X$ . Then the convolution operators with convolution kernels  $\gamma_l(t - s, \cdot)$  and  $\gamma_r(t - s, \cdot)$  are bounded on  $L^2(\mathbb{R}^+; X)$  (cf. Lemma 2.11). Then (a variation of) standard methods

(cf. [68, Sec. I.8], [69, Ch. XIII]) show that

$$\phi(t) = g(t) + \int_0^\infty \gamma(t, s, g(s)) ds,$$

where

$$\gamma(t, s, x) = \begin{cases} \gamma_r(t - s, x) + \int_0^s \gamma_r(t - \tau, \gamma_l(s - \tau, x)) d\tau, & 0 \leq s < t < \infty, \\ \gamma_l(s - t, x) + \int_0^t \gamma_r(t - \tau, \gamma_l(s - \tau, x)) d\tau, & 0 \leq t < s < \infty, \end{cases}$$

is the unique solution of (4.8) in  $L^2(\mathbb{R}^+; X)$ .

**(c)  $\implies$  (b)** Suppose  $\phi \in L^2(\mathbb{R}^+; X)$  is the unique solution of (4.8) for  $g = E(\cdot; S_0)x$ , where  $x \in X$ . Following the corresponding part of the proof of Theorem 4.3 but taking the scalar product with an arbitrary vector  $z \in X$ , we have

$$\begin{aligned} \langle \phi(t + u, x), z \rangle &= \int_0^\infty \langle E(t - s; S_0) \Gamma \phi(s + u, x), z \rangle ds \\ &= \langle \phi(t + u, x), z \rangle - \int_u^\infty \langle E(t + u - s; S_0) \Gamma \phi(s, x), z \rangle ds \\ &= \langle E(t + u; S_0)x, z \rangle + \int_0^u \langle E(t + u - s; S_0) \Gamma \phi(s, x), z \rangle ds \\ &= \langle E(u; S_0)x + (P) \int_0^u E(u - s; S_0) \Gamma \phi(s, x) ds, E(t; S_0)^* z \rangle \\ &= \langle E(u; S_0)x + (P) \int_0^\infty E(u - s; S_0) \Gamma \phi(s, x) ds, E(t; S_0)^* z \rangle \\ &= \langle \phi(u, x), E(t; S_0)^* z \rangle = \langle E(t; S_0) \phi(u, x), z \rangle, \end{aligned}$$

which, as before, implies the product rule

$$\phi(t + u, x) = \phi(t, \phi(u, x)), \quad t, u \in \mathbb{R}^+, x \in X.$$

Letting  $\Pi$  stand for the linear operator defined by  $\Pi x = \phi(0^+, x)$  for  $x \in X$ , we thus find that  $\Pi$  is a bounded projection on  $X$  whose kernel coincides with  $\text{Im } E(0^-; S_0)$ .

If  $y \in \mathcal{D}(S) = \mathcal{D}(S_0)$  (so that  $E(t; S)y \in \mathcal{D}(S)$ ), we compute, for  $t > 0$  and arbitrary  $z \in X$ ,

$$\begin{aligned} \langle E(t; S)y, z \rangle &= \int_0^\infty \langle E(t - s; S_0) \Gamma E(s; S)y, z \rangle ds \\ &= \langle E(t; S)y, z \rangle - \left( \int_0^t + \int_t^\infty \right) \frac{\partial}{\partial s} \langle E(t - s; S_0) E(s; S)y, z \rangle ds \\ &= \langle E(t; S)y, z \rangle - \langle E(0^+; S_0) E(t; S)y, z \rangle + \langle E(0^-; S_0) E(t; S)y, z \rangle \\ &\quad + \langle E(t; S_0) E(0^+; S)y, z \rangle = \langle E(t; S_0) E(0^+; S)y, z \rangle. \end{aligned}$$

Hence for all  $y \in \mathcal{D}(S_0) = \mathcal{D}(S)$  we have

$$\phi(t, E(0^+; S)y) = E(t; S)y.$$

By continuous extension we then get  $\text{Im } E(0^+; S) \subset \text{Im } \Pi$ .

The inclusion  $\text{Im } \Pi \subset \text{Im } E(0^+; S)$  follows as for Theorem 4.3.  $\square$

## 4.2 Block operators and Riccati equations

In this section we study exponentially dichotomous operators which have the following representation with respect to the decomposition  $X = X_0 \dot{+} X_1$  of the underlying complex Banach space  $X$ :

$$S_0 = \begin{pmatrix} -A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad S = \begin{pmatrix} -A_0 & D \\ Q & A_1 \end{pmatrix}. \quad (4.21)$$

Here  $-A_0$  and  $-A_1$  are the infinitesimal generators of exponentially decaying strongly continuous semigroups on  $X_0$  and  $X_1$ , respectively, while  $D : X_1 \rightarrow X_0$  and  $Q : X_0 \rightarrow X_1$  are bounded linear operators. We call an operator  $S$  of the form (4.21) a *block operator*. We study the decompositions  $\text{Im } E(0^\pm; S_0) \dot{+} \text{Im } E(0^\mp; S) = X$  and the block operator representations of the corresponding projections to arrive at solutions of certain algebraic Riccati equations.

The exponential dichotomy of block operators  $S$  of the type (4.21), where  $X_0 = X_1$  is a complex Hilbert space and  $D = Q^*$  is bounded, has been studied in [114, 115, 113], generalizing results where  $A_0$  and  $A_1$  are also assumed selfadjoint [3]. As in the present chapter, in [113] the block operator  $S$  is considered as a Hamiltonian operator associated with algebraic Riccati equations, but at the expense of imposing a regularity condition and analyticity requirements on the unperturbed bisemigroup because of the use of the expression (1.9) to construct the separating projection.

The results obtained in [114] have been applied to certain  $\lambda$ -rational boundary eigenvalue problems and those in [115] to the Dirac equation. In [111, 110, 112] quadratic numerical ranges of block operators on complex Hilbert spaces are studied in detail. Block operators on complex Hilbert spaces with unbounded entries and spectral factorization of a corresponding operator function have been studied in [121, 4].

In linear systems theory it is well known how to employ the spectral decomposition of block operators (or so-called Hamiltonian operators) to arrive at solutions of Riccati equations [99, 52, 107, 45]. However, in most publications on the subject the (indefinite) scalar product structure of the underlying Hilbert or Krein space is used in a seemingly essential way to arrive at solutions of Riccati equations with certain selfadjointness and/or positivity structures. On the other hand, the discretization of the nonlinear integral equations for the reflection and transmission coefficients in radiative transfer in planetary atmospheres or neutron transport in

nuclear reactors leads to Riccati equations [92, 93, 94], where the solutions satisfy positivity requirements (in the lattice sense). Similar Riccati equations arise from the analysis of 2-D continuous time Markov processes [80, 32, 31]. When cast in a functional setting, the underlying Banach space is  $L^1$ . In our opinion this justifies studying Riccati equations also in a Banach space setting.

### 4.2.1 Riccati equations in complex Banach spaces

For the block operators  $S_0$  and  $S$  defined by (4.21) we now prove that  $S$  is exponentially dichotomous if  $D = 0$ . In the same way we prove that  $S$  is exponentially dichotomous if  $Q = 0$ . We put

$$\Gamma_Q = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}. \quad (4.22)$$

**Lemma 4.7.** *Let  $-A_0$  and  $-A_1$  be the infinitesimal generators of exponentially decaying strongly continuous semigroups on  $X_0$  and  $X_1$ , respectively, and let  $Q \in \mathcal{L}(X_0, X_1)$ . Then the block operator*

$$S_Q = \begin{pmatrix} -A_0 & 0 \\ Q & A_1 \end{pmatrix}$$

*is exponentially dichotomous on  $X = X_0 \dot{+} X_1$ . Moreover,*

$$E(t; S_Q)x = E(t; S_0)x + \int_{-\infty}^{\infty} E(t - \tau; S_0)\Gamma_Q E(\tau; S_0)x \, d\tau. \quad (4.23)$$

*Proof.* For  $|\operatorname{Re} \lambda| \leq \varepsilon$  we have

$$(\lambda - S_Q)^{-1} = \begin{pmatrix} (\lambda + A_0)^{-1} & 0 \\ (\lambda - A_1)^{-1}Q(\lambda + A_0)^{-1} & (\lambda - A_1)^{-1} \end{pmatrix}.$$

For  $x = \begin{pmatrix} x_0 & x_1 \end{pmatrix}^T \in X$  we thus have

$$(\lambda - S_Q)^{-1}x = \int_{-\infty}^{\infty} e^{-\lambda t} E(t; S_Q)x \, dt,$$

where

$$E(t; S_Q)x = \begin{cases} \begin{pmatrix} e^{-tA_0} & 0 \\ -\int_t^{\infty} e^{(t-s)A_1} Q e^{-sA_0} \, ds & 0 \end{pmatrix}, & t > 0, \\ \begin{pmatrix} 0 & 0 \\ -\int_0^{\infty} e^{(t-s)A_1} Q e^{-sA_0} \, ds & -e^{tA_1} \end{pmatrix}, & t < 0. \end{cases} \quad (4.24)$$

Here the integrals have to be interpreted as the Bochner integrals

$$\int_{\max(t,0)}^{\infty} e^{(t-s)A_1} Q e^{-sA_0} x_0 ds, \quad x_0 \in X_0.$$

Since either semigroup involved is exponentially decaying and strongly continuous, also  $E(t; S_Q)$  is exponentially decaying and strongly continuous (except for a strong jump discontinuity at  $t = 0$ ). Consequently,  $S_Q$  is exponentially dichotomous. Moreover, since  $[(\lambda - S_0)^{-1} \Gamma_Q]^2 = 0$ , we obtain

$$\begin{aligned} (\lambda - S_Q)^{-1} &= [I_X - (\lambda - S_0)^{-1} \Gamma_Q]^{-1} (\lambda - S_0)^{-1} \\ &= [I_X + (\lambda - S_0)^{-1} \Gamma_Q] (\lambda - S_0)^{-1}, \end{aligned} \quad (4.25)$$

and therefore (4.23) holds.

It remains to prove that  $e^{\varepsilon|\cdot|} E(\cdot; S_Q) x \in L^\infty(\mathbb{R}; X)$  for some  $\varepsilon > 0$  and each  $x \in X$ . Indeed, from (4.25) we get

$$(\lambda - S_Q)^{-1} x = \int_{-\infty}^{\infty} e^{-\lambda t} F(t; x) dt, \quad x \in X,$$

where

$$F(t; x) = E(t; S_0) x + \int_{-\infty}^{\infty} E(t - \tau; S_0) \Gamma_Q E(\tau; S_0) x d\tau, \quad x \in X.$$

Since  $e^{\varepsilon|\cdot|} E(\cdot; S_0) x \in L^1(\mathbb{R}; X)$  for some  $\varepsilon > 0$  and every  $x \in X$ , we can apply Lemma 2.10 to prove that  $e^{\varepsilon|\cdot|} F(\cdot; x) \in L^1(\mathbb{R}; X)$  and hence that  $F(\cdot; x)$  is strongly measurable for every  $x \in X$ . From  $\|E(t; S_0)\| \leq M e^{-c|t|}$  for certain  $c, M > 0$ , we also get

$$\begin{aligned} \|F(t; x)\| &\leq M \|x\| \left( e^{-c|t|} + \|\Gamma_Q\| \int_{-\infty}^{\infty} e^{-c|t-\tau|} e^{-c|\tau|} d\tau \right) \\ &= M \|x\| e^{-c|t|} \left( 1 + \frac{1 + c|t|}{c} \|\Gamma_Q\| \right), \end{aligned}$$

which decays exponentially as  $t \rightarrow \pm\infty$ . Theorem 1.7 then implies that  $S_Q$  is exponentially dichotomous.  $\square$

In the following cases the results of Chapter 2 can be used to prove that the block operator  $S$  defined by (4.21) is exponentially dichotomous:

- a. There exists  $\varepsilon > 0$  such that

$$C_\varepsilon \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq \varepsilon\} \subset \rho(S) \quad (4.26)$$

while  $-A_0$  and  $-A_1$  generate exponentially decaying analytic (or immediately norm continuous) semigroups (cf. Theorems 2.4 and 2.5).

- b. The operator  $D$  is compact and  $S$  does not have imaginary eigenvalues. We then easily show that (4.26) holds for some  $\varepsilon > 0$ . The exponential dichotomy of  $S$  then follows from Lemma 4.7 and Theorem 2.3.
- c. There exists  $\varepsilon > 0$  such that (4.26) is true and  $(\lambda - S)^{-1}$  is bounded on  $C_\varepsilon$ , while  $X_0$  and  $X_1$  are complex Hilbert spaces (cf. Theorem 2.13).

It is clear that

$$E(0^+; S_0) = \begin{pmatrix} I_{X_0} & 0 \\ 0 & 0 \end{pmatrix}, \quad E(0^-; S_0) = \begin{pmatrix} 0 & 0 \\ 0 & -I_{X_1} \end{pmatrix}. \quad (4.27)$$

Further, for the operator defined by (4.24) we have

$$E(0^+; S_Q) = \begin{pmatrix} I_{X_0} & 0 \\ -Z & 0 \end{pmatrix}, \quad E(0^-; S_Q) = \begin{pmatrix} 0 & 0 \\ -Z & -I_{X_1} \end{pmatrix},$$

where the linear operator  $Z$  defined by

$$Zx_0 = \int_0^\infty e^{-tA_1} Q e^{-tA_0} x_0 dt, \quad x_0 \in X_0, \quad (4.28)$$

satisfies the Lyapunov equation

$$Z[\mathcal{D}(A_0)] \subset \mathcal{D}(A_1), \quad A_1 Z + Z A_0 = Q \text{ on } \mathcal{D}(A_0). \quad (4.29)$$

Since the spectra of  $-A_0$  and  $A_1$  do not intersect, the Lyapunov equation (4.29) has a unique solution [69, Theorem I.4.1] (also [135]).<sup>1</sup>

In the next theorem we relate the equivalent conditions of Theorem 4.3 to the solvability of a Riccati equation and the equivalent conditions of Theorem 4.4 to the solvability of another Riccati equation.

**Theorem 4.8.** *Let  $-A_0$  and  $-A_1$  be the infinitesimal generators of an exponentially decaying strongly continuous semigroup on the complex Banach spaces  $X_0$  and  $X_1$ , respectively, let  $Q \in \mathcal{L}(X_0, X_1)$  and  $D \in \mathcal{L}(X_1, X_0)$ , and let the block operator  $S$  defined by (4.21) be exponentially dichotomous. Then the decomposition (4.7) is true if and only if there exists a bounded solution  $\Pi_+ : X_1 \rightarrow X_0$  of the Riccati equation*

$$\Pi_+[\mathcal{D}(A_1)] \subset \mathcal{D}(A_0), \quad A_0 \Pi_+ + \Pi_+ A_1 + \Pi_+ Q \Pi_+ - D = 0 \text{ on } \mathcal{D}(A_1), \quad (4.30)$$

where  $\sigma(A_1 + Q \Pi_+)$  is contained in the open right half-plane. Similarly, the decomposition (4.14) is true if and only if there exists a bounded solution  $\Pi_- : X_0 \rightarrow X_1$  of the Riccati equation

$$\Pi_-[\mathcal{D}(A_0)] \subset \mathcal{D}(A_1), \quad A_1 \Pi_- + \Pi_- A_0 + \Pi_- D \Pi_- - Q = 0 \text{ on } \mathcal{D}(A_0), \quad (4.31)$$

where  $\sigma(A_0 + D \Pi_-)$  is contained in the open right half-plane. Such solutions  $\Pi_+$  and  $\Pi_-$  are unique when they exist.

<sup>1</sup>Applying these references requires writing the Lyapunov equation (4.29) in the form  $Z(\lambda + A_0)^{-1} - (\lambda - A_1)^{-1}Z = -(\lambda - A_1)^{-1}Q(\lambda + A_0)^{-1}$ .



Note that (4.31) reduces to the Lyapunov equation (4.29) if  $D = 0$ .

*Proof.* Let (4.14) be satisfied and let  $\Pi$  denote the projection of  $X$  onto  $\text{Im } E(0^-; S)$  along  $\text{Im } E(0^+; S_0)$ . In view of (4.27) there exists  $\Pi_+ \in \mathcal{L}(X_1, X_0)$  such that

$$\Pi = \begin{pmatrix} 0 & \Pi_+ \\ 0 & I_{X_1} \end{pmatrix}, \quad I_X - \Pi = \begin{pmatrix} I_{X_0} & -\Pi_+ \\ 0 & 0 \end{pmatrix}.$$

Because  $\text{Im } \Pi$  is an  $S$ -invariant subspace of  $X$ , there exists a linear operator  $B_1$  defined on a dense domain in  $X_1$  such that

$$\begin{pmatrix} -A_0 & D \\ Q & A_1 \end{pmatrix} \begin{pmatrix} \Pi_+ \\ I_{X_1} \end{pmatrix} = \begin{pmatrix} \Pi_+ \\ I_{X_1} \end{pmatrix} B_1. \quad (4.32)$$

Then  $\mathcal{D}(B_1) = \mathcal{D}(A_1)$  with  $B_1 = A_1 + Q\Pi_+$  and (4.30) is true, while  $B_1$  is similar to the restriction of  $S$  to  $\text{Im } E(0^-; S)$  and hence has its spectrum in the open right half-plane. Conversely, if (4.30) has a solution  $\Pi_+$  as indicated above, then the range of  $\begin{pmatrix} \Pi_+ \\ I_{X_1} \end{pmatrix}$  is an  $S$ -invariant subspace on which the restriction of  $S$  has its spectrum in the open right half-plane. Thus this range is necessarily contained in  $\text{Im } E(0^-; S)$  and has  $\text{Im } E(0^+; S_0)$  as its closed complement. Hence it must coincide with  $\text{Im } E(0^-; S)$ . Consequently, (4.14) is true.

Let (4.7) be satisfied and let  $\mathcal{Q}$  be the projection of  $X$  onto  $\text{Im } E(0^+; S)$  along  $\text{Im } E(0^-; S_0)$ . In view of (4.27) there exists  $\Pi_- \in \mathcal{L}(X_0, X_1)$  such that

$$\mathcal{Q} = \begin{pmatrix} I_{X_0} & 0 \\ -\Pi_- & 0 \end{pmatrix}, \quad I_X - \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ \Pi_- & I_{X_1} \end{pmatrix}.$$

Because  $\text{Im } \mathcal{Q}$  is an  $S$ -invariant subspace of  $X$ , there exists a linear operator  $B_0$  defined on a dense domain in  $X_0$  such that

$$\begin{pmatrix} -A_0 & D \\ Q & A_1 \end{pmatrix} \begin{pmatrix} I_{X_0} \\ -\Pi_- \end{pmatrix} = \begin{pmatrix} I_{X_0} \\ -\Pi_- \end{pmatrix} (-B_0). \quad (4.33)$$

Then  $\mathcal{D}(B_0) = \mathcal{D}(A_0)$  with  $B_0 = A_0 + D\Pi_-$  and (4.31) is true, while  $B_0$  is similar to the restriction of  $-S$  to  $\text{Im } E(0^+; S)$  and hence has its spectrum in the open right half-plane. Conversely, if (4.31) has a solution  $\Pi_-$  as indicated above, then the range of  $\begin{pmatrix} I_{X_0} \\ -\Pi_- \end{pmatrix}$  is an  $S$ -invariant subspace on which the restriction of  $-S$  has its spectrum in the open right half-plane. Then this range is necessarily contained in  $\text{Im } E(0^+; S)$  and has  $\text{Im } E(0^-; S_0)$  as its closed complement. Hence it must coincide with  $\text{Im } E(0^+; S)$ . Consequently, (4.7) is true.  $\square$

Let us now use the solutions  $\Pi_+$  and  $\Pi_-$  of the Riccati equations (4.30) and (4.31) (when they exist) to derive the left and right quasi-canonical factorization

of the operator function  $W$  given by (4.5) with respect to the imaginary axis. For  $|\operatorname{Re} \lambda| \leq \varepsilon$  we first write (4.30) in the form

$$-(\lambda + A_0)^{-1}\Pi_+ + \Pi_+(\lambda - A_1)^{-1} + (\lambda + A_0)^{-1}\Pi_+Q\Pi_+(\lambda - A_1)^{-1} - (\lambda + A_0)^{-1}D(\lambda - A_1)^{-1} = 0.$$

Then the left quasi-canonical factorization of  $W$  with respect to the imaginary axis is given by

$$W(\lambda) = \begin{pmatrix} I_{X_0} & -(\lambda + A_0)^{-1}D \\ 0 & I_{X_1} \end{pmatrix} \begin{pmatrix} W^l(\lambda) & 0 \\ 0 & I_{X_1} \end{pmatrix} \begin{pmatrix} I_{X_0} & 0 \\ -(\lambda - A_1)^{-1}Q & I_{X_1} \end{pmatrix},$$

where

$$W^l(\lambda) = [I_{X_0} + (\lambda + A_0)^{-1}\Pi_+Q] [I_{X_0} - \Pi_+(\lambda - A_1)^{-1}Q], \\ W^l(\lambda)^{-1} = [I_{X_0} + \Pi_+(\lambda - B_1)^{-1}Q] [I_{X_0} - (\lambda + \tilde{B}_0)^{-1}\Pi_+Q].$$

Here  $\tilde{B}_0 = A_0 + \Pi_+Q$  and  $B_1 = A_1 + Q\Pi_+$  are the infinitesimal generators of exponentially decaying strongly continuous semigroups on  $X_0$ . Analogously, writing the Riccati equation (4.31) in the form

$$-\Pi_-(\lambda + A_0)^{-1} + (\lambda - A_1)^{-1}\Pi_- + (\lambda - A_1)^{-1}\Pi_-D\Pi_-(\lambda + A_0)^{-1} - (\lambda - A_1)^{-1}Q(\lambda + A_0)^{-1} = 0,$$

we obtain the right quasi-canonical factorization of  $W$  with respect to the imaginary axis

$$W(\lambda) = \begin{pmatrix} I_{X_0} & 0 \\ -(\lambda - A_1)^{-1}Q & I_{X_1} \end{pmatrix} \begin{pmatrix} I_{X_0} & 0 \\ 0 & W^r(\lambda) \end{pmatrix} \begin{pmatrix} I_{X_0} & -(\lambda + A_0)^{-1}D \\ 0 & I_{X_1} \end{pmatrix},$$

where

$$W^r(\lambda) = [I_{X_1} - (\lambda - A_1)^{-1}\Pi_-D] [I_{X_1} + \Pi_-(\lambda + A_0)^{-1}D], \\ W^r(\lambda)^{-1} = [I_{X_1} - \Pi_1(\lambda + B_1)^{-1}D] [I_{X_1} + (\lambda - \tilde{B}_0)^{-1}\Pi_-D].$$

Here  $B_0 = A_0 + D\Pi_-$  and  $\tilde{B}_1 = A_1 + \Pi_-D$  are the infinitesimal generators of exponentially decaying strongly continuous semigroups on  $X_1$ .

When both of the decompositions (4.7) and (4.14) are valid, we have

$$S \begin{pmatrix} I_{X_0} & \Pi_+ \\ -\Pi_- & I_{X_1} \end{pmatrix} = \begin{pmatrix} I_{X_0} & \Pi_+ \\ -\Pi_- & I_{X_1} \end{pmatrix} \begin{pmatrix} -B_0 & 0 \\ 0 & B_1 \end{pmatrix}. \quad (4.34)$$

Here the block operator containing  $\Pi_+$  and  $\Pi_-$  is invertible on  $X$ , because  $\operatorname{Im} E(0^+; S) \dot{+} \operatorname{Im} E(0^-; S) = X$ . The invertibility of this block operator is equivalent to the invertibility of  $I + \Pi_+\Pi_-$  on  $X_0$  and to the invertibility of  $I + \Pi_-\Pi_+$  on  $X_1$ .

Consider the vector-valued Wiener-Hopf integral equations

$$u(t; x) - \int_0^\infty E(t - \tau; S_0) \Gamma u(\tau; x) d\tau = E(t; S_0)x, \quad t \in \mathbb{R}^+, \quad (4.35a)$$

$$v(t; x) - \int_{-\infty}^0 E(t - \tau; S_0) \Gamma v(\tau; x) d\tau = E(t; S_0)x, \quad t \in \mathbb{R}^-, \quad (4.35b)$$

where  $x \in \text{Im } E(0^+; S_0)$  and  $x \in \text{Im } E(0^-; S_0)$ , respectively. Using the classical Wiener-Hopf technique it is easily verified that

$$\begin{aligned} \hat{u}_+(\lambda; x) &= \int_0^\infty e^{-\lambda t} u(t; x) dt = \begin{pmatrix} (\lambda + B_0)^{-1} x_0 \\ -\Pi_- (\lambda + B_0)^{-1} x_0 \end{pmatrix}, \\ \hat{v}_-(\lambda; x) &= \int_{-\infty}^0 e^{-\lambda t} v(t; x) dt = \begin{pmatrix} \Pi_+ (\lambda - B_1)^{-1} x_1 \\ (\lambda - B_1)^{-1} x_1 \end{pmatrix}, \end{aligned}$$

where  $x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$ . Thus

$$u(t; x) = \begin{pmatrix} e^{-tB_0} x_0 \\ -\Pi_- e^{-tB_0} x_0 \end{pmatrix}, \quad v(t; x) = \begin{pmatrix} \Pi_+ e^{tB_1} x_1 \\ e^{tB_1} x_1 \end{pmatrix}.$$

Consequently, the solutions of the Riccati equations are given by

$$\Pi_+ x_1 = (0 \quad I_{X_0}) v(0^+; x), \quad \Pi_- x_0 = -(I_{X_0} \quad 0) u(0^-; x), \quad (4.36)$$

where  $x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$ .

#### 4.2.2 Riccati equations in complex Hilbert spaces

Now let  $X_0 = X_1$  be a complex Hilbert space,  $A_0 = A$ , and  $A_1 = A^*$ , where  $-A$  is the infinitesimal generator of a strongly continuous exponentially decaying semigroup on  $X_0$ . Let  $D$  and  $Q$  be positive selfadjoint operators. Then

$$S_0 = \begin{pmatrix} -A & 0 \\ 0 & A^* \end{pmatrix}, \quad S = \begin{pmatrix} -A & D \\ Q & A^* \end{pmatrix}. \quad (4.37)$$

Moreover,  $\Gamma = \Gamma_1 \Gamma_2$ , where

$$\Gamma_1 = \begin{pmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} D^{1/2} & 0 \\ 0 & Q^{1/2} \end{pmatrix}.$$

Consequently,

$$I_X + \Gamma_2 (\lambda - S_0)^{-1} \Gamma_1 = \begin{pmatrix} I_{X_0} & Q^{1/2} (\lambda - A)^{-1} D^{1/2} \\ D^{1/2} (\lambda + A^*)^{-1} Q^{1/2} & I_{X_0} \end{pmatrix}$$

has a positive selfadjoint real part for imaginary  $\lambda$ , i.e., (4.19) is satisfied for  $\delta = 1$ . As a result, all five statements (a)-(e) of Corollary 4.5 are true. In particular, we have the decompositions

$$\operatorname{Im} E(0^+; S) + \operatorname{Im} E(0^-; S_0) = \operatorname{Im} E(0^-; S) + \operatorname{Im} E(0^+; S_0) = X. \quad (4.38)$$

The following result is immediate from Theorem 4.8.

**Theorem 4.9.** *Let  $-A$  be the infinitesimal generator of an exponentially decaying strongly continuous semigroup on the complex Hilbert space  $X_0$ , let  $Q$  and  $D$  be nonnegative selfadjoint operators on  $X_0$ , and let the block operator  $S$  defined by (4.21) be exponentially dichotomous. Then there exists a bounded solution  $\Pi_+$  of the Riccati equation*

$$\Pi_+[\mathcal{D}(A^*)] \subset \mathcal{D}(A), \quad A\Pi_+ + \Pi_+A^* + \Pi_+Q\Pi_+ - D = 0 \text{ on } \mathcal{D}(A^*), \quad (4.39)$$

where  $\sigma(A + \Pi_+Q)$  is contained in the open right half-plane. Similarly, there exists a bounded solution  $\Pi_-$  of the Riccati equation

$$\Pi_-[\mathcal{D}(A)] \subset \mathcal{D}(A^*), \quad A^*\Pi_- + \Pi_-A + \Pi_-D\Pi_- - Q = 0 \text{ on } \mathcal{D}(A), \quad (4.40)$$

where  $\sigma(A + D\Pi_-)$  is contained in the open right half-plane. Such solutions  $\Pi_+$  and  $\Pi_-$  are selfadjoint and are unique.

The block operator  $S$  in (4.37) has interesting symmetry properties [73, 74]. Introducing the signature operators

$$J_1 = J_1^* = J_1^{-1} = \begin{pmatrix} 0 & iI_{X_0} \\ -iI_{X_0} & 0 \end{pmatrix}, \quad J_2 = J_2^* = J_2^{-1} = \begin{pmatrix} 0 & I_{X_0} \\ I_{X_0} & 0 \end{pmatrix},$$

we easily obtain

$$(J_1S)^* = -J_1S, \quad (4.41a)$$

$$\langle [J_2S + (J_2S)^*]x, x \rangle \geq 0, \quad (4.41b)$$

where  $x \in X$ . Equation (4.41a) implies that the maximal invariant subspaces of  $S$  on which  $S$  has its spectrum confined to the left or the right half-plane, is  $J_1$ -neutral in the sense that  $\langle J_1x, x \rangle = 0$  for  $x$  in such a subspace. This property implies that  $\Pi_+$  and  $\Pi_-$  are selfadjoint. Equation (4.41b) implies that

$$\langle J_2 \begin{pmatrix} \Pi_+ \\ I_{X_0} \end{pmatrix} x, \begin{pmatrix} \Pi_+ \\ I_{X_0} \end{pmatrix} x \rangle \geq 0, \quad \langle J_2 \begin{pmatrix} I_{X_0} \\ -\Pi_- \end{pmatrix} x, \begin{pmatrix} I_{X_0} \\ -\Pi_- \end{pmatrix} x \rangle \leq 0,$$

for any  $x \in X$ , which implies that  $\Pi_+$  and  $\Pi_-$  are positive selfadjoint operators.

### 4.3 Approximating solutions of Riccati equations

In this section we study the approximation of the solutions of Riccati equations by the solutions of finite-dimensional Riccati equations. In fact, starting from the block operators  $S_0$  and  $S$  defined by (4.37), where  $-A$  is the infinitesimal generator of an exponentially decaying strongly continuous semigroup on the complex Hilbert space  $X_0$ , we assume  $D$  to be a compact operator and employ a strong approximation of the identity operator on  $X_0$  by finite rank projections to arrive at an approximation of the solutions of the Riccati equations (4.39) and (4.40) by the solutions of the corresponding finite-dimensional Riccati equations, thus reproducing the results obtained in [38].

Finite-dimensional approximations of solutions of algebraic Riccati equations have been studied in many papers (e.g., [14, 89, 90, 125]). Compared to [90], we do not discuss the algebraic Riccati equation deriving from  $H^\infty$  control theory, but rather restrict ourselves to the one stemming from  $LQ$  optimal control theory. We basically obtain the same result as in [90], under somewhat different assumptions, but with a completely different proof. In [125] the algebraic Riccati equation derived from  $LQ$  optimal control was studied under much weaker assumptions than in this monograph (and in [38]), but under our assumptions we obtain much stronger convergence results than in [125].

Let  $X_n$  be a sequence of closed linear subspaces of  $X_0$ , not necessarily finite-dimensional, though in applications they usually are. Then there exist unique bounded linear operators  $\iota_n : X_n \rightarrow X_0$  and  $\pi_n : X_0 \rightarrow X_n$  such that  $\iota_n \pi_n$  is the orthogonal projection of  $X_0$  onto  $X_n$  and  $\pi_n \iota_n$  is the identity operator on  $X_n$ . We shall assume that  $\iota_n \pi_n$  tends to  $I_{X_0}$  in the strong sense.

Let us define the compressions  $D_n = \pi_n D \iota_n$  and  $Q_n = \pi_n Q \iota_n$ . Then  $D_n$  and  $Q_n$  are nonnegative selfadjoint operators on  $X_n$  whenever  $D$  and  $Q$  are nonnegative selfadjoint operators on  $X_0$ . Now let  $-A_n$  be the infinitesimal generator of an exponentially decaying strongly continuous semigroup on  $X_n$ . Then  $\theta_n = (A_n, Q_n, D_n; X_n)$  is called an *approximant* to the triple  $\theta = (A, Q, D; X_0)$  if for some  $\varepsilon > 0$  we have the approximation

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|\hat{\iota}_n E(t; S_{0n}) \hat{\pi}_n x - E(t; S_0)x\|_X = 0 \quad (4.42)$$

for each  $x \in X = X_0 \dot{+} X_0$ , uniformly in  $t \in \mathbb{R} \setminus \{0\}$ . Here  $\hat{\pi}_n = \pi_n \dot{+} \pi_n$ ,  $\hat{\iota}_n = \iota_n \dot{+} \iota_n$ , and  $S_{0n} = (-A_n) \dot{+} (A_n)^*$ . We remark that  $\iota_n Q_n \pi_n$  converges to  $Q$  strongly, while  $\iota_n D_n \pi_n$  converges to  $D$  in the operator norm, the latter because of the compactness of  $D$ .

**Theorem 4.10.** *Let  $\theta_n = (A_n, Q_n, D_n; X_n)$  be a sequence of approximants to the triple  $\theta = (A, Q, D; X_0)$ , where  $D$  is a compact operator. Put*

$$S_n = \begin{pmatrix} -A_n & D_n \\ Q_n & (A_n)^* \end{pmatrix}.$$

Then for some  $\varepsilon > 0$  we have

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|\hat{i}_n E(t; S_n) \hat{\pi}_n x - E(t; S)x\|_X = 0 \quad (4.43)$$

for each  $x \in X = X_0 \dot{+} X_0$ , uniformly in  $t \in \mathbb{R} \setminus \{0\}$ .

*Proof.* Put

$$S_n^Q = \begin{pmatrix} -A_n & 0 \\ Q_n & (A_n)^* \end{pmatrix}.$$

Then

$$E(t; S_n^Q)x = E(t; S_{0n})x - \int_{-\infty}^{\infty} E(t-s; S_{0n})\Gamma_{Q_n}E(s; S_n^Q)x ds, \quad (4.44)$$

where  $0 \neq t \in \mathbb{R}$ . Here

$$\Gamma_Q = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}, \quad \Gamma_{Q_n} = \begin{pmatrix} 0 & 0 \\ Q_n & 0 \end{pmatrix}.$$

Because  $\|E(t; S_{0n})\|$  has a finite upper bound which is independent of  $t \in \mathbb{R} \setminus \{0\}$  and  $n \in \mathbb{N}$ , we can apply the Dominated Convergence Theorem to the  $\theta_n$  analog

$$E(t; S_{Q_n})x = E(t; S_{0n})x + \int_{-\infty}^{\infty} E(t-\tau; S_{0n})\Gamma_{Q_n}E(\tau; S_{0n})x d\tau$$

of (4.44) to take the limit under the integral sign and prove that, for some  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|\hat{i}_n E(t; S_n^Q) \hat{\pi}_n x - E(t; S^Q)x\|_X = 0 \quad (4.45)$$

for each  $x \in X = X_0 \dot{+} X_0$ , uniformly in  $t \in \mathbb{R} \setminus \{0\}$ .

Next, consider the convolution equation

$$E(t; S_n)x = \int_{-\infty}^{\infty} E(t-\tau; S_n^Q)\Gamma_n^D E(\tau; S_n)x d\tau = E(t; S_n^Q)x,$$

where  $\Gamma_n^D = \begin{pmatrix} 0 & D_n \\ 0 & 0 \end{pmatrix}$ . This integral equation implies that

$$\hat{i}_n E(t; S_n) \hat{\pi}_n x - \int_{-\infty}^{\infty} \hat{i}_n E(t-\tau; S_n^Q) \Gamma_n^D \hat{i}_n E(\tau; S_n) \hat{\pi}_n x d\tau = \hat{i}_n E(t; S_n^Q) \hat{\pi}_n x, \quad (4.46)$$

where  $x \in X \dot{+} X$ . Note that  $\Gamma_n^D = \hat{\pi}_n \Gamma^D \hat{i}_n$  implies that we can rewrite the above equation in the form

$$\hat{i}_n E(t; S_n) \hat{\pi}_n x - \int_{-\infty}^{\infty} \hat{i}_n E(t-\tau; S_n^Q) \hat{\pi}_n \Gamma^D \cdot \hat{i}_n E(\tau; S_n) \hat{\pi}_n x d\tau = \hat{i}_n E(t; S_n^Q) \hat{\pi}_n x,$$

where  $x \in X \dot{+} X$ . Equation (4.45) and the compactness of  $\Gamma^D = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$  imply that, for some  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|\hat{\imath}_n E(t; S_n^Q) \hat{\pi}_n \Gamma^D - E(t; S^Q) \Gamma^D\|_{\mathcal{L}(X)} = 0$$

uniformly in  $t \in \mathbb{R} \setminus \{0\}$ . Since (4.46) is uniquely solvable on the complex Banach space  $BC(\mathbb{R}^-; X_n \dot{+} X_n) \dot{+} BC(\mathbb{R}^+; X_n \dot{+} X_n)$  of bounded continuous  $(X_n \dot{+} X_n)$ -valued functions on the real line with a possible strong jump discontinuity in  $t = 0$ , we get (4.43) with the help of (4.45), as claimed.  $\square$

To prove the strong stability of  $\Pi_-$  and the operator norm stability of  $\Pi_+$  on approximation of the triple  $\theta = (A, Q, D; X_0)$  by triples  $\theta_n = (A_n, Q_n, D_n; X_n)$ , we need to study the operator Wiener-Hopf equations (4.35), where  $x \in \text{Im } E(0^+; S_0)$  and  $x \in \text{Im } E(0^-; S_0)$ , respectively. The solutions of the Riccati equations are then given by (4.36). Analyzing (4.35) on  $BC(\mathbb{R}^\pm; X_0 \dot{+} X_0)$  is far from straightforward as, in general, the integral kernel  $E(\cdot; S_0)\Gamma$  is not Bochner integrable and hence (4.35) can seemingly only be studied effectively on  $L^2(\mathbb{R}^\pm; X_0 \dot{+} X_0)$ . To avoid doing so, we modify the integral kernel.

Let us introduce the modified operator convolution kernel

$$\begin{aligned} K(t; S_0) &= \begin{pmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{pmatrix} E(t; S_0) \begin{pmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} 0 & 0 \\ -D^{1/2} e^{tA^*} Q^{1/2} & 0 \end{pmatrix}, & t < 0, \\ \begin{pmatrix} 0 & Q^{1/2} e^{-tA} D^{1/2} \\ 0 & 0 \end{pmatrix}, & t > 0, \end{cases} \end{aligned}$$

which is compact and norm continuous in  $0 \neq t \in \mathbb{R}$ , as a result of the compactness of  $D^{1/2}$ . As a result,  $K$  is Bochner integrable. Furthermore, we have the intertwining property

$$\begin{pmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{pmatrix} E(t; S_0) \Gamma = K(t; S_0) \begin{pmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{pmatrix}.$$

Now consider the auxiliary Wiener-Hopf integral equations

$$w(t; x) - \int_0^\infty K(t - \tau; S_0) w(\tau; x) d\tau = \begin{pmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{pmatrix} E(t; S_0) x, \quad t \in \mathbb{R}^+, \quad (4.47a)$$

$$z(t; x) - \int_{-\infty}^0 K(t - \tau; S_0) z(\tau; x) d\tau = \begin{pmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{pmatrix} E(t; S_0) x, \quad t \in \mathbb{R}^-, \quad (4.47b)$$

where  $x \in \text{Im } E(0^+; S_0)$  and  $x \in \text{Im } E(0^-; S_0)$ , respectively. Equations (4.47) are uniquely solvable, because their (combined) symbol

$$\begin{aligned} I - \int_{-\infty}^{\infty} e^{-\lambda t} K(t) dt &= I - \begin{pmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{pmatrix} (\lambda - S_0)^{-1} \begin{pmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} I_{X_0} & -Q^{1/2}(\lambda - A)^{-1}D^{1/2} \\ -D^{1/2}(\lambda + A^*)^{-1}Q^{1/2} & I_{X_0} \end{pmatrix} \end{aligned}$$

has a positive real part for purely imaginary  $\lambda$  and hence has both a left and a right canonical factorization. From the solutions of (4.47) we find

$$u(t; x) = E(t; S_0)x + \int_0^{\infty} E(t - \tau; S_0) \begin{pmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{pmatrix} w(\tau; x) d\tau, \quad (4.48a)$$

$$v(t; x) = E(t; S_0)x + \int_{-\infty}^0 E(t - \tau; S_0) \begin{pmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{pmatrix} z(\tau; x) d\tau, \quad (4.48b)$$

where  $x \in \text{Im } E(0^{\pm}; S_0)$  and  $t \in \mathbb{R}^{\pm}$ , respectively. From (4.48), (4.47) is immediate.

Let us now consider a bounded solution  $\Pi_{+,n} : X_n \rightarrow X_n$  of the Riccati equation

$$\begin{cases} \Pi_{+,n}[\mathcal{D}((A_n)^*)] \subset \mathcal{D}(A_n), \\ A_n \Pi_{+,n} + \Pi_{+,n} (A_n)^* + \Pi_{+,n} Q_n \Pi_{+,n} - D_n = 0 \text{ on } \mathcal{D}((A_n)^*), \end{cases} \quad (4.49)$$

where  $\sigma(A_n + \Pi_{+,n} Q_n)$  is contained in the open right half-plane. Similarly, consider a bounded solution  $\Pi_{-,n} : X_n \rightarrow X_n$  of the Riccati equation

$$\begin{cases} \Pi_{-,n}[\mathcal{D}(A_n)] \subset \mathcal{D}((A_n)^*), \\ (A_n)^* \Pi_{-,n} + \Pi_{-,n} A_n + \Pi_{-,n} D_n \Pi_{-,n} - Q_n = 0 \text{ on } \mathcal{D}(A_n), \end{cases} \quad (4.50)$$

where  $\sigma(A_n + D_n \Pi_{-,n})$  is contained in the open right half-plane. Such solutions are unique when they exist.

We now derive the following strong convergence result for the solutions of the Riccati equations (4.39) and (4.40).

**Theorem 4.11.** *For each  $x \in X_0$  we have*

$$\lim_{n \rightarrow \infty} \|\iota_n \Pi_{-,n} \pi_n x - \Pi_- x\| = 0, \quad (4.51a)$$

$$\lim_{n \rightarrow \infty} \|\iota_n \Pi_{+,n} \pi_n x - \Pi_+ x\| = 0. \quad (4.51b)$$

*Proof.* Using the strong convergence  $\iota_n Q_n^{1/2} \pi_n \rightarrow Q^{1/2}$  and the compactness of  $D^{1/2}$ , we obtain for some  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|\hat{\iota}_n K(t; S_0) \hat{\pi}_n - K(t)\| = 0,$$



uniformly in  $0 \neq t \in \mathbb{R}$ . Hence for some  $\varepsilon > 0$  we have

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} e^{\varepsilon|t|} \|\hat{\imath}_n K(t; S_0) \hat{\pi}_n - K(t)\| dt = 0.$$

Letting  $w_n$  and  $z_n$  stand for the natural analogs of the solutions  $w$  and  $z$  of (4.47) and using the unique solvability of (4.47), we get for some  $\varepsilon > 0$  and each  $x \in X_0 \dot{+} X_0$ ,

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|\hat{\imath}_n w_n(t; \hat{\pi}_n x) - w(t; x)\| = 0,$$

uniformly in  $t \in \mathbb{R}^+$ . Similarly, for some  $\varepsilon > 0$  and every  $x \in X_0 \dot{+} X_0$  we have

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|\hat{\imath}_n z_n(t; \hat{\pi}_n x) - z(t; x)\| = 0,$$

uniformly in  $t \in \mathbb{R}^-$ . With the help of (4.48) we obtain for some  $\varepsilon > 0$  and each  $x \in X_0 \dot{+} X_0$ ,

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|\hat{\imath}_n u_n(t; \hat{\pi}_n x) - u(t; x)\| = 0,$$

uniformly in  $t \in \mathbb{R}^+$ . Similarly, for some  $\varepsilon > 0$  and every  $x \in X_0 \dot{+} X_0$  we have

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|\hat{\imath}_n v_n(t; \hat{\pi}_n x) - v(t; x)\| = 0,$$

uniformly in  $t \in \mathbb{R}^-$ . Equations (4.51) then follow using (4.36).  $\square$

In analogy with (4.28) (with  $A_0 = A$  and  $A_1 = A^*$ ) we define

$$Z_n x_0 = \int_0^\infty e^{-t(A_n)^*} Q_n e^{-tA_n} x_0 dt, \quad x_0 \in X_0, \quad (4.52)$$

which satisfies the Lyapunov equation

$$Z_n[\mathcal{D}(A_n)] \subset \mathcal{D}((A_n)^*), \quad (A_n)^* Z_n + Z_n A_n = Q_n \text{ on } \mathcal{D}(A_n). \quad (4.53)$$

Let us strengthen Theorem 4.11 and derive convergence properties in the operator norm.

**Theorem 4.12.** *We have*

$$\lim_{n \rightarrow \infty} \|\imath_n \Pi_{+,n} \pi_n - \Pi_+\| = 0, \quad (4.54a)$$

$$\lim_{n \rightarrow \infty} \|\imath_n (\Pi_{-,n} - Z_n) \pi_n - (\Pi_- - Z)\| = 0. \quad (4.54b)$$

*Proof.* We begin the proof by observing that the right-hand side of (4.47b) is given by  $0 \dot{+} D^{1/2} e^{tA^*} x_1$ , where  $t \in \mathbb{R}^+$  and  $D^{1/2}$  is compact. Moreover, as  $\theta_n$  converges to  $\theta$ , we have for some  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} e^{\varepsilon|t|} \|K(t; S_{0n}) - K(t; S_0)\| = 0,$$

uniformly in  $0 \neq t \in \mathbb{R}$ . Therefore

$$\lim_{n \rightarrow \infty} \|\hat{z}_n z_n(t; x) - z(t; x)\| = 0,$$

uniformly in  $t \in \mathbb{R}^+$  and in  $x_1$  on bounded subsets of  $X_0$ . Using (4.48b) and (4.36) we can then sharpen (4.51b) and derive (4.54a) instead.

By considering  $S_Q$  as the unperturbed exponentially dichotomous operator, we obtain instead of (4.35b), (4.47b), and (4.48b),

$$u_Q(t; x) - \int_0^\infty E(t - \tau; S_Q) \Gamma^D u_Q(\tau; x) d\tau = E(t; S_Q)x, \quad (4.55a)$$

$$w_Q(t; x) - \int_0^\infty K(t - \tau; S_Q) w_Q(\tau; x) d\tau = \begin{pmatrix} 0 & 0 \\ 0 & D^{1/2} \end{pmatrix} E(t; S_Q)x, \quad (4.55b)$$

$$u_Q(t; x) = E(t; S_Q)x + \int_0^\infty E(t - \tau; S_Q) \begin{pmatrix} 0 & D^{1/2} \\ 0 & 0 \end{pmatrix} w_Q(\tau; x) d\tau, \quad (4.55c)$$

where  $x \in X_0 \dot{+} X_0$ ,  $t \in \mathbb{R}^+$ ,  $\Gamma^D = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$ , and

$$\begin{pmatrix} 0 & 0 \\ 0 & D^{1/2} \end{pmatrix} E(t; S_Q) \Gamma^D = K(t; S_Q) \begin{pmatrix} 0 & 0 \\ 0 & D^{1/2} \end{pmatrix}.$$

Now note that, as a result of the compactness of  $D^{1/2}$ ,

$$\lim_{n \rightarrow \infty} \left\| \hat{z}_n \begin{pmatrix} 0 & 0 \\ 0 & D_n^{1/2} \end{pmatrix} E(t; S_{Q_n}) \hat{\pi}_n - \begin{pmatrix} 0 & 0 \\ 0 & D^{1/2} \end{pmatrix} E(t; S_Q) \right\| = 0,$$

uniformly in  $t \in \mathbb{R}^+$ . Consequently,

$$\lim_{n \rightarrow \infty} \|z_{Q_n}(t; \hat{\pi}_n x) - z_Q(t; x)\| = 0,$$

uniformly in  $t \in \mathbb{R}^+$  and in  $x$  on the unit ball of  $X_0 \dot{+} X_0$ . Repeating the proof of (4.54a) and using (4.55) we arrive at (4.54b).  $\square$

It remains to consider the approximation properties of the semigroups generated by  $-B_0 = -(A + D\Pi_-)$  and  $-B_1 = -(A^* + Q\Pi_+)$ , which are related to  $S$  by means of (4.34). It is easily verified that

$$\begin{pmatrix} I_{X_0} & \Pi_+ \\ -\Pi_- & I_{X_0} \end{pmatrix}^{-1} S \begin{pmatrix} I_{X_0} & \Pi_+ \\ -\Pi_- & I_{X_0} \end{pmatrix} = \begin{cases} e^{-t(A+D\Pi_-)} \dot{+} 0, & t > 0, \\ 0 \dot{+} (-e^{t(A^*+Q\Pi_+)}), & t < 0. \end{cases} \quad (4.56)$$

**Theorem 4.13.** *We have*

$$\lim_{n \rightarrow \infty} \left\| z_n e^{-t(A_n + D_n \Pi_{-,n})} \pi_n - e^{-t(A + D \Pi_-)} \right\| = 0, \quad (4.57a)$$

$$\lim_{n \rightarrow \infty} \left\| z_n e^{-t(A_n^* + Q_n \Pi_{+,n})} \pi_n - e^{-t(A^* + Q \Pi_+)} \right\| = 0, \quad (4.57b)$$

uniformly in  $t$  on compact subintervals of  $\mathbb{R}^+$ .

*Proof.* Put

$$M = \begin{pmatrix} I_{X_0} & \Pi_+ \\ -\Pi_- & I_{X_0} \end{pmatrix}, \quad M_n = \begin{pmatrix} I_{X_n} & \Pi_{+,n} \\ -\Pi_{-,n} & I_{X_n} \end{pmatrix}.$$

Then it suffices to prove that

$$\lim_{n \rightarrow \infty} \|\hat{i}_n M_n \hat{\pi}_n - M\|x\| = 0, \quad (4.58a)$$

$$\lim_{n \rightarrow \infty} \|\hat{i}_n M_n^{-1} \hat{\pi}_n - M^{-1}\|x\| = 0. \quad (4.58b)$$

Indeed, the invertibility of  $M$  implies the invertibility of  $I + \Pi_+ \Pi_-$  and  $I + \Pi_- \Pi_+$ , and

$$M^{-1} = \begin{pmatrix} (I + \Pi_+ \Pi_-)^{-1} & -(I + \Pi_+ \Pi_-)^{-1} \Pi_+ \\ (I + \Pi_- \Pi_+)^{-1} \Pi_- & (I + \Pi_- \Pi_+)^{-1} \end{pmatrix}.$$

Since

$$\lim_{n \rightarrow \infty} \|\hat{i}_n (I + \Pi_{-,n} \Pi_{+,n}) \hat{\pi}_n - (I + \Pi_- \Pi_+)\| = 0,$$

and similarly with  $\Pi_+$  and  $\Pi_-$  interchanged, we have

$$\lim_{n \rightarrow \infty} \|\hat{i}_n (I + \Pi_{-,n} \Pi_{+,n})^{-1} \hat{\pi}_n - (I + \Pi_- \Pi_+)^{-1}\| = 0,$$

and similarly with  $\Pi_+$  and  $\Pi_-$  interchanged. As a consequence we get (4.58), as claimed.  $\square$

# Chapter 5

## Transport Equations

Linear transport equations in plane-parallel homogeneous media have been studied as abstract boundary value problems on complex Hilbert spaces for three decades [82, 83, 24, 15, 152, 102, 77]. Here we study their evolution operators as multiplicative perturbations of exponentially dichotomous operators, first for multiplicative perturbations that are compact perturbations of the identity, then for positive selfadjoint (bounded as well as unbounded) multiplicative perturbations. We also derive formal solutions of the relevant boundary value problems.

### 5.1 Introduction

Stationary neutron transport, radiative transfer and rarefied gas dynamics, as well as the linearized Boltzmann equation with a hard or Maxwellian potential, in spatially homogeneous plane-parallel media are described by a linear integrodifferential equation with partial range boundary conditions [40, 43, 147, 42]. These boundary value problems can be written in the following abstract form. Given an injective selfadjoint operator  $T$  on a complex Hilbert space  $H$  with scalar product  $\langle \cdot, \cdot \rangle$ , let  $Q_{\pm}$  denote the orthogonal projections onto the maximal closed subspaces  $H_{\pm}$  of  $H$  on which  $\langle \pm Tx, x \rangle \geq 0$ . Then by the abstract kinetic boundary value problem we mean the vector-valued differential equation

$$(T\psi)'(x) = -A\psi(x), \quad 0 < x < \tau, \quad (5.1)$$

with the boundary conditions

$$\begin{cases} \|Q_+\psi(x) - \varphi_+\|_H = 0, & x \rightarrow 0^+, \\ \|\psi(x)\|_H = O(1), & x \rightarrow +\infty, \end{cases} \quad (5.2)$$

if  $\tau = +\infty$ , and

$$\begin{cases} \|Q_+\psi(x) - \varphi_+\|_H = 0, & x \rightarrow 0^+, \\ \|Q_-\psi(x) - \varphi_-\|_H = 0, & x \rightarrow \tau^-, \end{cases} \quad (5.3)$$

if  $\tau$  is finite. Here  $A$  is a compact perturbation of the identity (i.e.,  $A = I_H - B$ , where  $B$  is a compact operator). For reasons that will be apparent shortly, we call  $A$  the (*abstract*) *collision operator* and  $T$  the (*abstract*) *streaming operator*.

The classical example in radiative transfer [43, 147] occurs for isotropic scattering and albedo of single scattering  $a \in (0, 1]$ . Writing the specific intensity as a function  $\psi(x, \mu)$  depending on position  $x \in (0, \tau)$  (in optical length) and direction cosine  $\mu \in [-1, 1]$ , we have the boundary value problem

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \psi(x, \mu) = \frac{a}{2} \int_{-1}^1 \psi(x, \mu') d\mu', \quad x \in (0, \tau), \quad \mu \in [-1, 1], \quad (5.4)$$

with boundary conditions

$$\begin{cases} \psi(0, \mu) = \varphi_+(\mu), & 0 \leq \mu \leq 1, \\ \left[ \int_{-1}^1 |\psi(x, \mu)|^2 d\mu \right]^{1/2} = O(1), & x \rightarrow +\infty, \end{cases} \quad (5.5)$$

for  $\tau = +\infty$ , and boundary conditions

$$\begin{cases} \psi(0, \mu) = \varphi_+(\mu), & 0 \leq \mu \leq 1, \\ \psi(\tau, \mu) = \varphi_-(\mu), & -1 \leq \mu \leq 0, \end{cases} \quad (5.6)$$

for finite  $\tau$ . The boundary conditions specify the specific intensity of the incident radiation. Putting  $H = L^2(-1, 1)$  and defining the streaming and collision operators by

$$(Tf)(\mu) = \mu f(\mu), \quad (Af)(\mu) = f(\mu) - \frac{a}{2} \int_{-1}^1 f(\mu') d\mu',$$

where  $\mu \in [-1, 1]$ , and the orthogonal projections  $Q_\pm$  by

$$(Q_\pm f)(\mu) = \begin{cases} f(\mu), & (\pm\mu) > 0, \\ 0, & (\pm\mu) < 0, \end{cases}$$

we obtain from (5.4)–(5.6) the boundary value problems (5.1)–(5.3).

Equations (5.4)–(5.6) also arise in neutron transport theory [39, 40, 158], where the above operators  $T$ ,  $A$ , and  $Q_\pm$  were first introduced in [82]. Now  $\psi(x, \mu)$  stands for the neutron density as a function of position  $x \in (0, \tau)$  (in neutron mean free path) and direction cosine  $\mu \in [-1, 1]$ . Here the interaction between the neutrons and the background medium is assumed isotropic and  $a > 0$  is the

average neutron production per collision. In fact, in locally supercritical media we always have  $a > 1$ .

In the scalar BGK model of rarefied gas dynamics [41, 42], the deviation of the particle distribution  $\psi(x, v)$  from the Maxwellian equilibrium distribution is assumed to depend on position  $x \in (0, \tau)$  (in gas molecule mean free path) and velocity  $v \in \mathbb{R}$ . For various specific problems such as Couette flow and Poiseuille flow, we have the boundary value problem

$$v \frac{\partial \psi}{\partial x}(x, v) + \psi(x, v) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi(x, v') e^{-(v')^2} dv', \quad x \in (0, \tau), \quad v \in [-1, 1], \quad (5.7)$$

with boundary conditions

$$\begin{cases} \psi(0, v) = \varphi_+(v), & v \geq 0, \\ \left[ \int_{-\infty}^{\infty} |\psi(x, v)|^2 e^{-v^2} dv \right]^{1/2} = O(1), & x \rightarrow +\infty, \end{cases} \quad (5.8)$$

for  $\tau = +\infty$ , and boundary conditions

$$\begin{cases} \psi(0, v) = \varphi_+(v), & v \geq 0, \\ \psi(\tau, v) = \varphi_-(v), & v \leq 0, \end{cases} \quad (5.9)$$

for finite  $\tau$ . Putting  $H = L^2(\mathbb{R}, \pi^{-1/2} e^{-v^2} dv)$  and defining the streaming and collision operators by

$$(Tf)(v) = vf(v), \quad (Af)(v) = f(v) - \pi^{-1/2} \int_{-\infty}^{\infty} f(v') e^{-(v')^2} dv',$$

where  $v \in \mathbb{R}$ , and the orthogonal projections  $Q_{\pm}$  by

$$(Q_{\pm}f)(v) = \begin{cases} f(v), & (\pm v) > 0, \\ 0, & (\pm v) < 0, \end{cases}$$

we obtain from (5.7)–(5.9) the boundary value problems (5.1)–(5.3). For the scalar BGK equation the operators  $T$ ,  $A$ , and  $Q_{\pm}$  were first introduced in [100].

Neutron transport, radiative transfer and rarefied gas dynamics lead to a major class of integrodifferential equations with boundary conditions which can be put in the form of (5.1)–(5.3). In neutron transport one may deal with groups of neutrons of different speeds, anisotropic interactions, and interactions between different groups of neutrons, which in general leads to a nonselfadjoint integral operator  $B$ . In radiative transfer anisotropic scattering leads to selfadjoint and contractive integral operators  $B$  if polarization of light is not taken into account or, more generally, if linear and circular polarization effects do not interact [87]. The integral operator  $B$  is generally nonselfadjoint with respect to the natural

scalar product of the underlying Hilbert space but selfadjoint in a suitable indefinite scalar product [155, 77]. In rarefied gas dynamics, apart from the few rather peculiar multispecies BGK models that generally lead to nonselfadjoint integral operators  $B$ , assuming a hard or Maxwellian (binary) interaction between the gas molecules and linearizing the nonlinear Boltzmann equation about the equilibrium solution, leads to the above type of boundary value problem. Details on converting practical stationary transport equations into boundary value problems of the form (5.1)–(5.3) can be found in [102, 77]. For the non BGK type models of rarefied gas dynamics we refer to [42].

Equation (5.1) is an evolution equation of the type

$$\psi'(x) = -T^{-1}A\psi(x), \quad 0 < x < \tau.$$

When studying this problem on the half-line ( $\tau = +\infty$ ), the boundary condition in (5.2) suggests treating  $-T^{-1}A$  as an exponentially dichotomous operator on the complex Hilbert space, so that the solution of (5.1) with boundary condition (5.2) has the form

$$\psi(x) = E(x; -T^{-1}A)\psi(0),$$

where  $Q_+\psi(0) = \varphi_+$ . Instead, the solution of (5.1) with boundary condition (5.3) has the form

$$\psi(x) = [E(x; -T^{-1}A) - E(x - \tau; -T^{-1}A)]\chi,$$

where

$$\begin{aligned} Q_+[E(0^+; -T^{-1}A) - E(-\tau; -T^{-1}A)]\chi &= \varphi_+, \\ Q_-[-E(0^-; -T^{-1}A) + E(\tau; -T^{-1}A)]\chi &= \varphi_-. \end{aligned}$$

In other words, we need to establish if  $-T^{-1}A$  is exponentially dichotomous and, if this is the case, to solve  $\varphi_{\pm}$  from the matching conditions (5.2) or (5.3).

Since  $T$  is an injective selfadjoint operator on  $H$ , the operator  $-T^{-1}$  is exponentially dichotomous with separating projection  $Q_- = I - Q_+$  if (and only if)  $T$  is bounded. The unbounded operator  $-T^{-1}A$  may therefore be viewed as a multiplicative perturbation of  $-T^{-1}$  obtained by postmultiplying it by a compact (additive) perturbation of the identity. Thus we cannot rely on the perturbation theory of exponentially dichotomous operators expounded in Chapter 2, but instead we need to use similar methods to arrive at the relevant perturbation results.

In Section 5.3 we shall consider the boundary value problem (5.1)–(5.3) in a general context. For a better understanding of the operator theory involved, we shall now briefly discuss the **finite-dimensional case** in which  $T$  is a nonsingular hermitian  $n \times n$  matrix and  $A$  an  $n \times n$  matrix. This finite-dimensional case is at the basis of the discrete ordinates method to solve the transport equation numerically by discretizing the angular variables [163, 146, 145]. It has been studied for its own sake in [133, 71]. Let  $Q_{\pm}$  be the Riesz projection of  $T$  corresponding to its eigenvalues in  $\mathbb{R}^{\pm}$ , and let  $P_+$ ,  $P_-$ , and  $P_0$  be the Riesz projections of  $T^{-1}A$

corresponding to its eigenvalues in the open right half-plane, the open left half-plane, and the imaginary axis, respectively.

When  $\tau$  is finite, the general solution of (5.1) has the form

$$\psi(x) = e^{-xT^{-1}A}\phi_p + e^{(\tau-x)T^{-1}A}\phi_m + e^{-xT^{-1}A}\phi_0,$$

where  $\phi_p \in \text{Im } P_+$ ,  $\phi_m \in \text{Im } P_-$ , and  $\phi_0 \in \text{Im } P_0$ . Matching the boundary conditions (5.3) leads to the identity

$$V_\tau \phi = \varphi \stackrel{\text{def}}{=} \varphi_+ + \varphi_-,$$

where

$$V_\tau \stackrel{\text{def}}{=} Q_+ \left[ P_+ + e^{\tau T^{-1}A} P_- \right] + Q_- \left[ P_- + e^{-\tau T^{-1}A} P_+ + e^{-\tau T^{-1}A} P_0 \right].$$

The unique solvability of the boundary value problem for all  $\varphi_\pm \in \text{Im } Q_\pm$  is equivalent to the invertibility of  $V_\tau$ .

When  $\tau = +\infty$ , the general solution of (5.1) has the form

$$\psi(x) = e^{-x^{-1}A}\phi$$

for some vector  $\phi$ . The boundedness condition as  $x \rightarrow +\infty$  implies

$$\phi \in H_{p+} \stackrel{\text{def}}{=} \text{Im } P_+ \dot{+} \bigoplus_{\text{Im } \lambda = 0} \text{Ker } (T^{-1}A - \lambda), \quad Q_+ \phi = \varphi_+.$$

Thus  $\varphi_+ = \phi - Q_- \phi \in H_{p+} + H_-$ , where  $H_- = \text{Im } Q_-$ . Hence the boundary value problem is uniquely solvable for every  $\varphi_+ \in \text{Im } Q_+$  if and only if the decomposition

$$H_{p+} \dot{+} H_- = \mathbb{C}^n$$

holds.

When discussing the boundary value problems in Section 5.3, we shall exploit these basic ideas. More involved methods are required to prove the invertibility of  $V_\tau$  or the above decomposition of the underlying Hilbert space when we are no longer dealing with the rather artificial finite-dimensional case.

## 5.2 Exponential dichotomy in transport theory

In this section we prove that  $-T^{-1}A$  is exponentially dichotomous and in fact generates an analytic bisemigroup if  $T$  is bounded, injective and selfadjoint,  $A$  is a compact perturbation of the identity, and  $T^{-1}A$  does not have zero or imaginary eigenvalues. We also prove that the difference between the bisemigroups,  $E(t; -T^{-1}A) - E(t; -T^{-1})$ , is a compact operator for every  $t \in \mathbb{R}$ , albeit by imposing the regularity condition (5.22) below. We pay special attention to the important special case where  $A$  is positive selfadjoint.



### 5.2.1 Preliminary results

Let  $T$  be an injective selfadjoint operator on a complex Hilbert space  $H$ . Let  $Q_{\pm}$  be the orthogonal projections of  $H$  onto the maximal  $T$ -invariant subspaces on which  $\pm\langle Tx, x \rangle \geq 0$ . For  $x \in H$  put

$$E(t; -T^{-1})x = \begin{cases} e^{-tT^{-1}}Q_+x = \int_0^{\infty} e^{-t/z}\sigma(dz)x, & t > 0, \\ -e^{-tT^{-1}}Q_-x = -\int_{-\infty}^0 e^{-t/z}\sigma(dz)x, & t < 0, \end{cases} \quad (5.10)$$

where  $\sigma(\cdot)$  stands for the resolution of the identity of the selfadjoint operator  $T$ . Then  $E(\cdot; -T^{-1})$  is strongly continuous except for a strong jump discontinuity at  $t = 0$ , vanishes in the strong operator topology as  $t \rightarrow \pm\infty$ ,  $E(0^{\pm}; -T^{-1}) = \pm Q_{\pm}$  (so that the size of the jump at  $t = 0$  equals  $I_H$ ), and

$$\begin{cases} E(t+s; -T^{-1}) = E(t; -T^{-1})E(s; -T^{-1}), & t, s > 0, \\ E(t+s; -T^{-1}) = -E(t; -T^{-1})E(s; -T^{-1}), & t, s < 0. \end{cases}$$

Hence,  $E(\cdot; -T^{-1})$  has all of the properties of a strongly continuous (and in fact analytic) bisemigroup on  $H$  except for its exponential decay as  $t \rightarrow \pm\infty$ . Only for  $T$  bounded the operator  $-T^{-1}$  is exponentially dichotomous and in fact generates an analytic bisemigroup.

For  $x \in H$  we now define

$$\mathcal{H}(t)x = \begin{cases} T^{-1}e^{-tT^{-1}}Q_+x = \int_0^{\infty} z^{-1}e^{-t/z}\sigma(dz)x, & t > 0, \\ -T^{-1}e^{-tT^{-1}}Q_-x = -\int_{-\infty}^0 z^{-1}e^{-t/z}\sigma(dz)x, & t < 0. \end{cases} \quad (5.11)$$

Then  $\mathcal{H}(t) = -(d/dt)E(t; -T^{-1})$  for any  $t \in \mathbb{C}$  with  $\operatorname{Re} t \neq 0$ , where the differentiation can be performed in the operator norm. Then for all vectors  $x, y \in X$  we have

$$\begin{aligned} \int_{-\infty}^{\infty} \langle \mathcal{H}(t)x, y \rangle dt &= \int_{-\infty}^0 \int_{-\infty}^0 \frac{-1}{t} e^{-z/t} dt \langle \sigma(dx)x, y \rangle \\ &\quad + \int_0^{\infty} \int_0^{\infty} \frac{1}{t} e^{-z/t} dt \langle \sigma(dx)x, y \rangle \\ &= \left( \int_{-\infty}^0 + \int_0^{\infty} \right) \langle \sigma(dz)x, y \rangle = \langle x, y \rangle, \end{aligned} \quad (5.12)$$

where we have used Fubini's theorem. Since the complex Hilbert space  $H$  can be represented as a direct integral of  $L^2$ -spaces so that  $T$  becomes a direct integral of operators of multiplication by the independent variable [23], there is a natural way to turn  $H$  into a complex Banach lattice and to construct from each  $x \in X$

an absolute value  $|x|$  having the same norm. Since the measure  $\langle \sigma(\cdot)|x|, |y| \rangle$  is nonnegative for any  $x, y \in \mathcal{H}$ , we have

$$\int_{-\infty}^{\infty} |\langle \mathcal{H}(t)x, y \rangle| dt \leq \int_{-\infty}^{\infty} \langle \mathcal{H}(t)|x|, |y| \rangle dt = \langle |x|, |y| \rangle \leq \|x\| \|y\|, \quad (5.13)$$

where  $x, y \in X$ . Hence,  $\mathcal{H}$  is a weakly integrable operator function as observed before in [62].

For any  $\alpha \in \mathbb{R}^+$  we now define  $|T|^\alpha \mathcal{H}(t)$  as follows:

$$|T|^\alpha \mathcal{H}(t)x = \begin{cases} |T|^{\alpha-1} e^{-tT^{-1}} Q_+ x = \int_0^\infty |z|^{\alpha-1} e^{-t/z} \sigma(dz)x, & t > 0, \\ |T|^{\alpha-1} e^{-tT^{-1}} Q_- x = \int_{-\infty}^0 |z|^{\alpha-1} e^{-t/z} \sigma(dz)x, & t < 0. \end{cases}$$

**Proposition 5.1.** *Let  $\alpha \in [0, 1)$ . Then*

$$\| |T|^\alpha \mathcal{H}(t) \| \leq (1 - \alpha)^{1-\alpha} e^{-(1-\alpha)} |t|^{\alpha-1}, \quad 0 \neq t \in \mathbb{R}.$$

*If  $T$  is bounded, we have*

$$\| |T|^\alpha \mathcal{H}(t) \| \leq \|T\|^{\alpha-1} e^{-t/\|T\|}, \quad |t| \geq (1 - \alpha)\|T\|.$$

*Proof.* Compute the maximum of the function  $|z|^{\alpha-1} e^{-|t|/|z|}$ . □

Now let  $B$  be a compact operator on  $H$  and define  $A = I - B$ . Then

$$\begin{aligned} W(\lambda) &= I - (I - \lambda T)^{-1} B = (I - \lambda T)^{-1} (A - \lambda T) \\ &= (\lambda - T^{-1})^{-1} (\lambda - T^{-1} A) \end{aligned} \quad (5.14)$$

is defined for all nonreal  $\lambda$  and, if  $T$  is bounded, also in a neighborhood of  $\lambda = 0$ . Since  $\|(I - \lambda T)^{-1} x - x\| \rightarrow 0$  as  $\lambda \rightarrow 0$  in the double cone  $K_\delta = \{\lambda \in \mathbb{C} : |\arg(\lambda) - \frac{\pi}{2}| < \delta\} \setminus \{0\}$  for any  $\delta \in (0, \frac{\pi}{2})$ , we have, in view of the compactness of  $B$ , that

$$\lim_{\lambda \rightarrow 0, \lambda \in K_\delta} \|W(\lambda) - A\| = 0. \quad (5.15)$$

On the other hand,  $\|(I - \lambda T)^{-1} x\| \rightarrow 0$  as  $|\lambda| \rightarrow +\infty$  in  $K_\delta$  for any  $\delta \in (0, \frac{\pi}{2})$ . Thus

$$\lim_{|\lambda| \rightarrow \infty, \lambda \in K_\delta} \|W(\lambda) - I_H\| = 0. \quad (5.16)$$

As a result, there are only finitely many nonzero points on the imaginary axis, where  $W(\lambda)$  is not invertible. If these do not occur, there exists  $\delta \in (0, \frac{\pi}{2})$  such that  $W(\lambda)$  is invertible for  $\lambda \in K_\delta$ .

Since the points  $0 \neq \lambda \in \mathbb{C}$  such that  $(1/\lambda) \notin \sigma(T)$  and  $W(\lambda)$  is not invertible, are exactly the eigenvalues of  $T^{-1}A$  (or  $AT^{-1}$ ) outside  $\sigma(T^{-1}) \setminus \{0\}$ , we have the following result.

**Proposition 5.2.** *The eigenvalues of  $T^{-1}A$  outside  $\sigma(T^{-1}) \setminus \{0\}$  have finite algebraic multiplicity and do not accumulate within any of the double cones  $K_\delta$  ( $\delta \in (\frac{\pi}{2}, \pi)$ ). Hence,  $T^{-1}A$  has at most finitely many imaginary eigenvalues.*

Proposition 5.2 implies that  $W(\lambda)$  is invertible for  $|\arg(\lambda) - \frac{\pi}{2}| \leq \delta_1$  for some  $\delta_1 \in (0, \frac{\pi}{2})$  if  $T^{-1}A$  does not have zero or imaginary eigenvalues. Equation (5.14) then implies that

$$\|(\lambda - T^{-1}A)^{-1}\| = O(1/|\lambda|), \quad |\lambda| \rightarrow +\infty, \quad |\arg(\lambda) - \frac{\pi}{2}| \leq \delta_1.$$

Thus  $T^{-1}A$  is bisectorial and hence, as a result of Proposition 1.8, generates an analytic bisemigroup.

We have thus proved the following.

**Theorem 5.3.** *Suppose  $T^{-1}A$  does not have zero or purely imaginary eigenvalues and  $T$  is bounded. Then  $-T^{-1}A$  generates an analytic bisemigroup.*

Suppose  $T$  is bounded. Then the zero and purely imaginary eigenvalues of  $T^{-1}A$  are isolated and have a finite algebraic multiplicity. Since there are only finitely many of them, the eigenvectors and generalized eigenvectors of  $T^{-1}A$  corresponding to zero and purely imaginary eigenvalues span a finite-dimensional subspace  $H_0$  of  $H$ . Letting  $\Gamma$  be a simple positively oriented Jordan contour encircling the zero and purely imaginary eigenvalues of  $T^{-1}A$  once and no other point of the spectrum of  $T^{-1}A$ , there exists a finite rank projection

$$P_0 = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T^{-1}A)^{-1} d\lambda = \frac{-1}{2\pi i} \int_{\Gamma} (A - \lambda T)^{-1} T d\lambda$$

of  $H$  such that  $\text{Im } P_0$  and  $\text{Ker } P_0$  are both invariant under  $T^{-1}A$  and

$$\begin{aligned} \sigma\left(T^{-1}A|_{\text{Im } P_0}\right) &= \{\lambda \in \sigma(T^{-1}A) : \text{Re } \lambda = 0\}, \\ \sigma\left(T^{-1}A|_{\text{Ker } P_0}\right) &= \{\lambda \in \sigma(T^{-1}A) : \text{Re } \lambda \neq 0\}. \end{aligned}$$

Now let  $\beta$  be a linear operator on  $\text{Im } P_0$  without zero or purely imaginary eigenvalues, and let

$$A_\beta = A(I - P_0) + T\beta^{-1}P_0. \quad (5.17)$$

Then

$$T^{-1}A_\beta = T^{-1}A|_{\text{Ker } P_0} \dot{+} \beta^{-1}$$

has the property that  $B_\beta \stackrel{\text{def}}{=} I - A_\beta$  is compact and  $T^{-1}A_\beta$  does not have zero or purely imaginary eigenvalues. According to Theorem 5.3,  $-T^{-1}A_\beta$  generates an analytic bisemigroup. Consequently, the restriction of  $-T^{-1}A$  to  $\text{Ker } P_0$  generates an analytic bisemigroup.

### 5.2.2 Positive selfadjoint collision operators

In the example (5.4)–(5.6) with albedo of single scattering  $a \in (0, 1)$  we have an operator  $A$  which is positive selfadjoint. This situation occurs in many radiative transfer equations (with polarization not taken into account or where linear and circular polarization effects do not interact) or neutron transport equations (with one neutron speed only) if there is net absorption in the medium.

If  $A$  is a positive selfadjoint operator, the usual scalar product of  $H$  is equivalent to the scalar product

$$\langle x, y \rangle_A = \langle Ax, y \rangle = \langle A^{1/2}x, A^{1/2}y \rangle, \quad x, y \in H, \quad (5.18)$$

while  $A^{-1}T$  and hence  $T^{-1}A$  are selfadjoint with respect to this scalar product. Let  $P_{\pm}$  the orthogonal (with respect to (5.18)) projections on  $H$  onto the maximal  $A^{-1}T$ -subspaces on which  $\pm \langle A^{-1}Tx, x \rangle_A \geq 0$ . Put

$$E(t; -T^{-1}A)x = \begin{cases} e^{-tT^{-1}A}P_+x = \int_0^\infty e^{-t/z}\tilde{\sigma}(dz)x, & t > 0, \\ -e^{-tT^{-1}A}P_-x = -\int_{-\infty}^0 e^{-t/z}\tilde{\sigma}(dz)x, & t < 0, \end{cases} \quad (5.19)$$

where  $\tilde{\sigma}(\cdot)$  stands for the resolution of the identity of the selfadjoint operator  $A^{-1}T$ . Then  $E(\cdot; -T^{-1}A)$  is strongly continuous except for a strong jump discontinuity at  $t = 0$ , vanishes in the strong operator topology as  $t \rightarrow \pm\infty$ ,  $E(0^\pm; -T^{-1}A) = \pm P_{\pm}$  (so that the size of the jump at  $t = 0$  equals  $I_H$ ), and

$$\begin{cases} E(t+s; -T^{-1}A) = E(t; -T^{-1}A)E(s; -T^{-1}A), & t, s > 0, \\ E(t+s; -T^{-1}A) = -E(t; -T^{-1}A)E(s; -T^{-1}A), & t, s < 0. \end{cases}$$

Hence,  $E(\cdot; -T^{-1}A)$  has all of the properties of a strongly continuous (and in fact analytic) bisemigroup on  $H$  except for its exponential decay as  $t \rightarrow \pm\infty$ . Only for  $T$  (and hence  $A^{-1}T$ ) bounded the operator  $-T^{-1}A$  is exponentially dichotomous and in fact generates an analytic bisemigroup.

Now suppose  $A$  is nonnegative selfadjoint but has a nontrivial kernel. We have the following simple result [152, 153, 77], which can also be derived from Proposition 6.1 in the next chapter.

**Proposition 5.4.** *Let  $A$  be nonnegative selfadjoint and  $T$  bounded. Then  $T^{-1}A$  does not have purely imaginary eigenvalues. Further, if  $(T^{-1}A)^n x = 0$  for some  $n \in \mathbb{N}$  and  $x \in H$ , then  $(T^{-1}A)^2 x = 0$ .*

*Proof.* Suppose  $Ax = \lambda Tx$ , where  $\operatorname{Re} \lambda = 0$ . Then

$$0 \leq \langle Ax, x \rangle = \lambda \langle Tx, x \rangle.$$

Then either  $\lambda \neq 0$  and  $\langle Ax, x \rangle = \langle Tx, x \rangle = 0$ , or  $\lambda = 0$  and  $\langle Ax, x \rangle = 0$ . In the former case we have  $0 = \langle Ax, x \rangle = \|A^{1/2}x\|^2$  and therefore  $Tx = (1/\lambda)Ax =$

$(1/\lambda)A^{1/2}(A^{1/2}x) = 0$  and hence  $x = 0$ . Consequently,  $T^{-1}A$  does not have purely imaginary eigenvalues.

In the latter case, suppose  $x, y, z \in H$  are such that

$$Ax = Ty, \quad Ay = Tz, \quad Az = 0.$$

Then

$$\|A^{1/2}y\|^2 = \langle Ay, y \rangle = \langle Tz, y \rangle = \langle z, Ty \rangle = \langle z, Ax \rangle = \langle Az, x \rangle = 0,$$

implying that  $Tz = Ay = A^{1/2}(A^{1/2}y) = 0$  and hence  $z = 0$ . Therefore,  $Ax = Ty$  and  $Ay = 0$ . Consequently,

$$\text{Ker}(T^{-1}A)^3 = \text{Ker}(T^{-1}A)^2,$$

which completes the proof.  $\square$

Defining  $H_0$  as in Subsection 5.2.1, we have

$$H_0 = \text{Ker}(T^{-1}A)^2.$$

Actually, in the example of (5.4)–(5.6), where  $H = L^2(-1, 1)$  and  $(Af)(\mu) = f(\mu) - \frac{a}{2} \int_{-1}^1 f(\mu') d\mu'$ , we have  $A$  invertible unless  $a = 1$ . In that case

$$H_0 = \{c_1 + c_2\mu : c_1, c_2 \in \mathbb{C}\}$$

is a two-dimensional subspace of  $L^2(-1, 1)$ .

### 5.2.3 Identity plus compact collision operator

Suppose  $T$  is bounded and  $A = I - B$  is a compact additive perturbation of the identity. Assume  $T^{-1}A$  does not have zero or purely imaginary eigenvalues. Then  $-T^{-1}A$  generates an analytic bisemigroup. Applying the inverse Laplace transform to (5.14) we obtain the vector-valued integral equation

$$E(t, -T^{-1}A)x + \int_{-\infty}^{\infty} \mathcal{H}(t - \tau)BE(\tau; -T^{-1}A)x d\tau = E(t; -T^{-1})x, \quad (5.20)$$

where  $0 \neq t \in \mathbb{R}$  and  $x \in H$ . Then the exponential dichotomy of  $-T^{-1}$  and  $-T^{-1}A$  implies that there exists  $\rho > 0$  such that, for  $c \in [-\rho, \rho]$ , both  $e^{c(\cdot)}E(\cdot; -T^{-1})x$  and  $e^{c(\cdot)}E(\cdot; -T^{-1}A)x$  belong to  $L^2(\mathbb{R}, H)$  for any  $x \in H$ . Thus, for  $c \in (-\rho, \rho)$ , both  $e^{c(\cdot)}E(\cdot; -T^{-1})x$  and  $e^{c(\cdot)}E(\cdot; -T^{-1}A)x$  belong to  $L^1(\mathbb{R}, H)$  for any  $x \in H$ .

**Proposition 5.5.** *Let  $T$  be bounded and suppose  $T^{-1}A$  does not have zero or purely imaginary eigenvalues. Then for any  $0 \neq t \in \mathbb{R}$  the linear operators*

$$E(t; -T^{-1}A) - E(t; -T^{-1})$$

*are compact.*

*Proof.* Write (5.20) in the form

$$D(t)x \stackrel{\text{def}}{=} E(t, -T^{-1}A)x - E(t; -T^{-1})x = - \int_{-\infty}^{\infty} \mathcal{H}(t - \tau) B E(\tau; -T^{-1}A)x d\tau,$$

where  $0 \neq t \in \mathbb{R}$  and  $x \in H$ . Note that the expressions on either side belong to  $L^1(\mathbb{R}, H)$ . Applying the Fourier transform we have

$$\int_{-\infty}^{\infty} e^{-\lambda t} D(t) dt = -(I - \lambda T)^{-1} B W(\lambda)^{-1} (I - \lambda T)^{-1} T, \quad (5.21)$$

where the integral in the left-hand side exists and is absolutely convergent in the operator norm. This easily follows from the fact that  $-T^{-1}$  and  $-T^{-1}A$  generate analytic bisemigroups on  $H$ . Further, the expression obtained is a compact operator irrespective of the choice of imaginary  $\lambda$ .

Let  $\mathcal{K}(H)$  denote the Banach space of all compact operators on  $H$ , endowed with the usual operator norm. Then  $\mathcal{K}(H)$  is a closed two-sided ideal in  $\mathcal{L}(H)$  and  $\mathcal{L}(H)/\mathcal{K}(H)$  is a Banach algebra called the Calkin algebra. Projecting (5.21) onto the Calkin algebra and applying an arbitrary continuous linear functional  $\Phi$  on  $\mathcal{L}(H)/\mathcal{K}(H)$  to the projected (5.21) we obtain

$$\int_{-\infty}^{\infty} e^{-\lambda t} \Phi([D(t)]) dt = 0, \quad \Phi \in (\mathcal{L}(H)/\mathcal{K}(H))^*,$$

where  $[D] \stackrel{\text{def}}{=} D + \mathcal{K}(H) \in \mathcal{L}(H)/\mathcal{K}(H)$ . Observe that  $\Phi$  can be moved inside the integral, because the integral in the left-hand side of (5.21) is a Bochner integral with respect to the operator norm. Also, the integrand  $\Phi([D(\cdot)])$  is continuous except for a strong jump continuity in  $t = 0$ . Consequently,  $\Phi([D(t)]) = 0$  for  $0 \neq t \in \mathbb{R}$ , irrespective of the choice of  $\Phi$ . But this means that  $D(t)$  is a compact operator for all  $0 \neq t \in \mathbb{R}$ , as claimed.  $\square$

The argument of the above proof does not go through if  $t = 0^\pm$ . Thus if  $Q_\pm = \pm E(0^\pm; -T^{-1})$  and  $P_\pm = \pm E(0^\pm; -T^{-1}A)$ , it is by no means clear if  $P_+ - Q_+$  is a compact operator. However, the integrals

$$A^{-1}TE(t; -T^{-1}A) - TE(t; -T^{-1}) = \begin{cases} \int_t^\infty D(\tau) d\tau, & t \geq 0, \\ -\int_{-\infty}^t D(\tau) d\tau, & t \leq 0, \end{cases}$$

which are absolutely convergent Bochner integrals with respect to the operator norm, show that, when applied for  $t = 0^\pm$ ,  $A^{-1}TP_+ - TQ_+$  is a compact operator. But then also  $T[P_+ - Q_+]$  is a compact operator.

The compactness of  $P_+ - Q_+$  can be proved under an additional regularity constraint as introduced in [151, 152] (also [77]).

**Proposition 5.6.** *Let  $T$  be bounded and suppose  $T^{-1}A$  does not have zero or purely imaginary eigenvalues. Assume in addition that*

$$\exists \alpha \in (0, 1) : \operatorname{Im} B \subset \operatorname{Im} |T|^\alpha. \quad (5.22)$$

*Then  $P_+ - Q_+$  is a compact operator.*

*Proof.* Condition (5.22) implies that  $B = |T|^\alpha D$  for some  $\alpha \in (0, 1)$  and  $D \in \mathcal{L}(H)$ . According to Proposition 5.1, we now have

$$\|\mathcal{H}(t)B\| = \begin{cases} O(|t|^{\alpha-1}), & t \rightarrow 0, \\ O(e^{-|t|/\|T\|}), & t \rightarrow \pm\infty. \end{cases}$$

Hence,

$$\int_{-\infty}^{\infty} \|\mathcal{H}(t)B\| dt < \infty. \quad (5.23)$$

Now note that

$$I + \int_{-\infty}^{\infty} e^{-\lambda t} \mathcal{H}(t)B dt = I - (I - \lambda T)^{-1}B = W(\lambda), \quad \operatorname{Re} \lambda = 0,$$

which is invertible for each zero or imaginary  $\lambda$  (including  $\lambda = \pm i\infty$ ). According to the Allan-Bochner-Phillips Theorem 2.1, there exists a Bochner integrable function  $F \in L^1(\mathbb{R}, \mathcal{L}(H))$  such that

$$W(\lambda)^{-1} = I - \int_{-\infty}^{\infty} e^{-\lambda t} F(t) dt, \quad \operatorname{Re} \lambda = 0.$$

Thus

$$E(t; -T^{-1}A) - E(t; -T^{-1}) = \int_{-\infty}^{\infty} F(t - \tau) E(\tau; -T^{-1}A) d\tau, \quad t \in \mathbb{R}.$$

Using that  $F$  can be approximated by integrable  $\mathcal{K}(\mathcal{H})$ -values step functions, where  $\mathcal{K}(\mathcal{H})$  is the Banach space of compact linear operators on  $H$  endowed with the usual operator norm, it follows that  $E(t; -T^{-1}A) - E(t; -T^{-1})$  is compact for any  $t \in \mathbb{R}$ , i.e., also for  $t = 0^\pm$ . As a result,  $P_+ - Q_+$  is a compact operator on  $H$ .  $\square$

If  $T$  is bounded and Condition (5.22) is satisfied, but  $T^{-1}A$  may have zero or imaginary eigenvalues, we choose  $\beta$  as in (5.17), where  $\beta$  does not have zero or imaginary eigenvalues. Since the projection  $P_0$  has finite rank, the operator  $B_\beta = I - A_\beta$  satisfies (5.22), provided  $\operatorname{Im} P_0 \subset \operatorname{Im} |T|^\alpha$  for some  $\alpha \in (0, 1)$ . However, if  $(T^{-1}A - \lambda)x_0 = 0$  for some  $n \in \mathbb{N}$  and  $\operatorname{Re} \lambda = 0$ , there exist  $x_0, x_1, \dots, x_{n-1} \in H$  such that

$$x_j = T(\lambda x_j + x_{j+1}) + Bx_j \quad (j = 0, 1, n-2), \quad x_{n-1} = (\lambda T + B)x_{n-1},$$

so that  $\{x_0, \dots, x_{n-1}\} \subset \mathcal{D}(|T|^\alpha)$  for some  $\alpha \in (0, 1)$ . Hence,  $\text{Im } P_0 \subset \mathcal{D}(|T|^\alpha)$  for some  $\alpha \in (0, 1)$ , as claimed.

Now observe that

$$E(t; -T^{-1}A_\beta) = E(t; -T^{-1}A|_{\text{Ker } P_0})(I - P_0) + e^{-t\beta^{-1}}P_0.$$

Putting  $P_\pm^{(\beta)} \stackrel{\text{def}}{=} \pm E(0^\pm; -T^{-1}A_\beta)$ , we see that  $P_\pm^{(\beta)} - Q_\pm$  as well as the difference  $E(t; -T^{-1}A_\beta) - E(t; -T^{-1})$  for each  $t \in \mathbb{R}$  are compact operators. Letting  $P_\pm$  stand for the projections  $P_\pm \stackrel{\text{def}}{=} P_\pm^{(\beta)}(I - P_0)$  which do not depend on  $\beta$ , we see that  $P_+ - Q_+$  is a compact operator on  $H$ . We now define

$$E(t; -T^{-1}A) \stackrel{\text{def}}{=} E(t; -T^{-1}A_\beta)(I - P_0) + e^{-T^{-1}A|_{\text{Im } P_0} P_0},$$

which does not depend on  $\beta$ . Then  $E(t; -T^{-1}A) - E(t; -T^{-1})$  is a compact operator for every  $t \in \mathbb{R}$ .

## 5.3 Solving the boundary value problems

In this section we solve the boundary value problems (5.1)–(5.3) by reduction to certain direct sum decompositions (for  $\tau = +\infty$ ) or to the invertibility of a linear operator (for  $\tau$  finite). Throughout this section  $T$  is bounded,  $A$  is a compact perturbation of the identity,  $B = I - A$ , and Condition (5.22) is satisfied.

### 5.3.1 Boundary value problems on a finite interval

First assume  $T^{-1}A$  does not have zero or imaginary eigenvalues. Then the solution of (5.1)–(5.3) (for finite  $\tau$ ) is given by

$$\psi(x) = [E(x; -T^{-1}A) - E(x - \tau; -T^{-1}A)] \phi$$

for some  $\phi \in H$ , where  $V_\tau \phi = \varphi$  for  $\varphi = \varphi_+ + \varphi_-$ . Here

$$V_\tau \phi \stackrel{\text{def}}{=} Q_+ [P_+ + e^{\tau T^{-1}A} P_-] \phi + Q_- [P_- + e^{-\tau T^{-1}A} P_+] \phi = \varphi.$$

If  $T^{-1}A$  has zero or imaginary eigenvalues, we have

$$\psi(x) = [E(x; -T^{-1}A) - E(x - \tau; -T^{-1}A)] (I - P_0) \phi + e^{-xT^{-1}A|_{\text{Im } P_0} P_0} \phi$$

for some  $\phi \in H$ , where  $V_\tau \phi = \varphi$ . Here

$$V_\tau \stackrel{\text{def}}{=} Q_+ [P_+ + e^{\tau T^{-1}A} P_- + P_0] + Q_- [P_- + e^{-\tau T^{-1}A} P_+ + e^{-\tau T^{-1}A} P_0].$$

Thus the boundary value problem (5.1)–(5.3) (for finite  $\tau$ ) is uniquely solvable if and only  $V_\tau$  is invertible.

The next theorem implies that the boundary value problem (5.1)–(5.3) is uniquely solvable (for finite  $\tau$ ) if  $A$  is nonnegative selfadjoint.



**Theorem 5.7.** *Let  $A$  be nonnegative selfadjoint. Then  $V_\tau$  is invertible. Thus the boundary value problem (5.1)–(5.3) is uniquely solvable for every  $\varphi_\pm \in \text{Im } Q_\pm$ .*

*Proof.* Since  $H_0 = \text{Ker } (T^{-1}A)^2$ , we have

$$V_\tau = Q_+ \left[ P_+ + e^{\tau T^{-1}A} P_- + P_0 \right] + Q_- \left[ P_- + e^{-\tau T^{-1}A} P_+ + (I - \tau T^{-1}A) P_0 \right].$$

Put  $H_\pm = \text{Im } Q_\pm$ ,  $H_p = \text{Im } P_+$ , and  $H_m = \text{Im } P_-$ . Suppose  $V_\tau \phi = 0$ , where  $\phi = \phi_p + \phi_m + \phi_0$ . Because of the selfadjointness of  $T^{-1}A$  in the (degenerate) scalar product (5.18), the vectors  $\phi_p$ ,  $\phi_m$ , and  $\phi_0$  are orthogonal also in the indefinite scalar product  $[x, y]_T = \langle Tx, y \rangle$  (for  $x, y \in H$ ). From  $\phi \in \text{Ker } V_\tau$  it follows that

$$[P_+ + e^{\tau T^{-1}A} P_- + P_0] \phi \in H_-, \quad [P_- + e^{-\tau T^{-1}A} P_+ + (I - \tau T^{-1}A) P_0] \phi \in H_+.$$

implying that

$$\begin{aligned} \langle T \phi_p, \phi_p \rangle + \langle T e^{2\tau T^{-1}A} P_- \phi_m, \phi_m \rangle + \langle T \phi_0, \phi_0 \rangle &\leq 0, \\ \langle T \phi_m, \phi_m \rangle + \langle T e^{-2\tau T^{-1}A} P_- \phi_p, \phi_p \rangle + \langle T (I - \tau T^{-1}A) \phi_0, \phi_0 \rangle &\geq 0. \end{aligned}$$

Subtracting these two equations we obtain

$$\begin{aligned} &\langle A^{-1} T [P_+ - e^{-2\tau T^{-1}A} P_+] \phi_p, \phi_p \rangle_A \\ &\quad + \langle A^{-1} T [P_- - e^{2\tau T^{-1}A} P_-] \phi_m, \phi_m \rangle_A + \tau \langle A \phi_0, \phi_0 \rangle \\ &= \langle T [P_+ - e^{-2\tau T^{-1}A} P_+] \phi_p, \phi_p \rangle + \langle T [P_- - e^{2\tau T^{-1}A} P_-] \phi_m, \phi_m \rangle \\ &\quad + \tau \langle A \phi_0, \phi_0 \rangle \leq 0, \end{aligned}$$

where each term in the left-hand side is nonnegative. Since each such term vanishes, we obtain  $\phi_p = \phi_m = A \phi_0 = 0$ . Thus  $\phi \in \text{Ker } A$  and therefore  $0 = V_\tau \phi = [Q_+ P_0 + Q_- (I - \tau T^{-1}A) P_0] \phi = \phi$ , which implies that  $\text{Ker } V_\tau = \{0\}$ .

To prove that  $I - V_\tau$  is compact, we compute

$$\begin{aligned} V_\tau - I &= (Q_+ - Q_-)(P_+ - Q_+) + Q_+[e^{\tau T^{-1}A} - e^{\tau T^{-1}A} Q_-] \\ &\quad + Q_-[e^{-\tau T^{-1}A} P_+ - e^{-\tau T^{-1}A} Q_+] + P_0 - \tau Q_- T^{-1} A P_0, \end{aligned}$$

which is easily seen to be compact as a result of Propositions 5.5 and 5.6 and  $\dim \text{Im } P_0 < \infty$ . Consequently,  $V_\tau$  is invertible.  $\square$

### 5.3.2 Boundary value problems on the half-line

First assume  $T^{-1}A$  does not have zero or imaginary eigenvalues. Then the solution of (5.1)–(5.3) (for finite  $\tau$ ) is given by

$$\psi(x) = E(x; -T^{-1}A)\phi,$$

where  $\phi \in \text{Im } P_+$  and  $Q_+\phi = \varphi_+$ . If  $T^{-1}A$  has zero or imaginary eigenvalues, we have

$$\psi(x) = E(x; -T^{-1}A)(I - P_0)\phi + \sum_{\substack{\lambda \in \sigma(T^{-1}A) \\ \text{Re } \lambda = 0}} e^{-\lambda x} \phi_\lambda,$$

where  $\phi_\lambda \in \text{Ker}(T^{-1}A - \lambda)$  and

$$Q_+ \left( \phi + \sum_{\substack{\lambda \in \sigma(T^{-1}A) \\ \text{Re } \lambda = 0}} \phi_\lambda \right) = \varphi_+.$$

**Theorem 5.8.** *The boundary value problem (5.1)–(5.3) (for  $\tau = +\infty$ ) has a unique solution if and only if*

$$\left[ \text{Im } P_+ \dot{+} \bigoplus_{\substack{\lambda \in \sigma(T^{-1}A) \\ \text{Re } \lambda = 0}} \text{Ker}(T^{-1}A - \lambda) \right] \dot{+} \text{Im } Q_- = H. \quad (5.24)$$

*Proof.* Let  $H_{p+}$  denote the subspace in (5.24) between square brackets and let  $H_\pm = \text{Im } Q_\pm$ . Then all solutions of the boundary value problem have the form

$$\psi(x) = E(x; -T^{-1}A)\phi, \quad (5.25)$$

where  $\phi \in H_{p+}$  and  $Q_+\phi = \varphi_+$ . Thus  $\varphi_+ = \phi - Q_-\phi \in [H_{p+} + H_-] \cap H_+$ . Moreover, the solutions of the corresponding homogeneous boundary value problem (where  $\varphi_+ = 0$ ) have the form (5.25), where  $\psi(0) \in H_{p+} \cap H_-$ . Since

$$\frac{H}{H_{p+} + H_-} = \frac{[H_{p+} + H_-] + H_+}{H_{p+} + H_-} \simeq \frac{H_+}{[H_{p+} + H_-] \cap H_+}$$

in the sense of vector space isomorphism, we see that the solution exists for any  $\varphi_+ \in H_+$  if and only if  $H_{p+} + H_- = H$ , and that there exists at most one solution if and only if  $H_{p+} \cap H_- = \{0\}$ .  $\square$

Suppose  $T^{-1}A$  does not have zero or purely imaginary eigenvalues. Put  $H_p = \text{Im } P_+$  and  $H_m = \text{Im } P_-$ . Then the necessary and sufficient condition for the unique solvability of (5.1)–(5.3) for every  $\varphi_+ \in H_+$  is that

$$H_p \dot{+} H_- = H.$$

Similarly, the necessary and sufficient condition for the unique solvability of its counterpart on the negative half-line is

$$H_m \dot{+} H_+ = H.$$

Now consider the linear operator

$$V = Q_+P_+ + Q_-P_- . \quad (5.26)$$

With the help of Proposition 5.6 we easily prove

**Lemma 5.9.** *Suppose  $T^{-1}A$  does not have zero or purely imaginary eigenvalues. Then*

$$\begin{aligned} \text{Ker } V &= [H_p \cap H_-] \dot{+} [H_m \cap H_+] , \\ \text{Im } V &= [H_p + H_-] \cap [H_m + H_+] . \end{aligned}$$

Thus  $V$  is invertible if and only if

$$H_p \dot{+} H_- = H = H_m \dot{+} H_+ . \quad (5.27)$$

Further,  $I - V = Q_+P_- + Q_-P_+ = (Q_- - Q_+)(P_+ - Q_+)$  is a compact operator.

As a result, the invertibility of  $V$  is equivalent to the simultaneous unique solvability of (5.1)–(5.3) and its counterpart for  $x \in \mathbb{R}^-$ . If  $A$  is positive selfadjoint and hence (5.18) defines an equivalent scalar product in  $H$  with respect to which  $A^{-1}T$  is selfadjoint, then (5.27) is satisfied.

Indeed, if  $x \in H_p \cap H_-$ , we have

$$\begin{aligned} x \in H_p \cap H_- &\implies \underbrace{\langle Tx, x \rangle}_{\leq 0, \text{ because } x \in H_-} = \underbrace{\langle A^{-1}Tx, x \rangle_A}_{\geq 0, \text{ because } x \in H_p} , \\ x \in H_m \cap H_+ &\implies \underbrace{\langle Tx, x \rangle}_{\geq 0, \text{ because } x \in H_+} = \underbrace{\langle A^{-1}Tx, x \rangle_A}_{\leq 0, \text{ because } x \in H_m} , \end{aligned}$$

whence  $\langle Tx, x \rangle = 0$  for  $x \in H_p \cap H_-$  and  $x \in H_m \cap H_+$ , which implies that  $x = 0$ . Since  $H_p \cap H_- = H_m \cap H_+ = \{0\}$  implies that  $\text{Ker } V = \{0\}$ , we get the invertibility of  $V$  and therefore the decompositions (5.27) with the help of the compactness of  $I - V$ , as claimed.

The auxiliary operator  $V$  and the above proof of the injectivity of  $V$  have been introduced in [83] when studying (5.4)–(5.6) for  $a \in (0, 1)$ . The unique solvability results for nonnegative definite  $A$  can be found in [151, 152, 77]. The more involved results for the boundary value problem on the half-line if  $A$  is non-strictly positive selfadjoint, are not given here.

## 5.4 Avoiding compactness assumptions

In [24] a method of studying the boundary value problem (5.1)–(5.3) for non-negative selfadjoint  $A$  has been developed without assuming any compactness of  $B = I - A$ . Instead, it suffices to assume that  $T$  is injective and selfadjoint and

$A$  is bounded and positive selfadjoint and has a closed range. The price to pay is that the boundary value problems are solved in an extension of the given complex Hilbert space  $H$ , namely in the completion  $H_T$  of the domain  $\mathcal{D}(T)$  of  $T$  with respect to the scalar product

$$\langle x, y \rangle_T = \langle |T|x, y \rangle, \quad x, y \in \mathcal{D}(T). \quad (5.28)$$

### 5.4.1 Bounded collision operators with closed range

Assuming  $A$  to be *bounded* positive selfadjoint with closed range, we follow the method used in [77] with some simplifications. When adapted to the present situation, the following result appeared for the first time in [105], but also in [116]. We present the concise proof given in [28, Theorem 1.2].

**Lemma 5.10.** *Suppose  $T$  is an injective selfadjoint operator on a complex Hilbert space  $H$ . Let  $K$  and  $\tilde{K}$  be bounded linear operators on  $H$  satisfying  $K[\mathcal{D}(T)] \subset \mathcal{D}(T)$  and*

$$TKx = \tilde{K}Tx, \quad x \in \mathcal{D}(T).$$

*Then  $K$  and  $\tilde{K}$  extend to bounded linear operators on  $H_T$ .*

*Proof.* For  $x, y \in \mathcal{D}(T)$  we have

$$\begin{aligned} \langle Kx, y \rangle_T &= \langle |T|Kx, y \rangle = \langle (Q_+ - Q_-)TKx, y \rangle \\ &= \langle (Q_+ - Q_-)\tilde{K}Tx, y \rangle = \langle Tx, \tilde{K}^*(Q_+ - Q_-)y \rangle \\ &= \langle x, \underbrace{(Q_+ - Q_-)\tilde{K}^*(Q_+ - Q_-)}_{\text{written as } K^{[*]}}y \rangle_T. \end{aligned}$$

Put  $s_n = \|(KK^{[*]})^n x\|_T^2$  for  $n = 0, 1, 2, \dots$  and  $x \in \mathcal{D}(T)$  with  $\|x\|_T = 1$ . Using the symmetry of  $KK^{[*]}$  with respect to (5.28), we get for  $\lambda \in \mathbb{R}$ ,

$$0 \leq \|(KK^{[*]})^{n-1}x + \lambda(KK^{[*]})^{n+1}x\|_T^2 = s_{n-1} + 2\lambda s_n + \lambda^2 s_{n+1}.$$

Thus the discriminant of this quadratic polynomial in  $\lambda$  is nonpositive and therefore  $s_0 = 1$  and

$$(s_n)^2 \leq s_{n-1}s_{n+1}, \quad n \in \mathbb{N}.$$

Hence,

$$s_1 = \frac{s_1}{s_0} \leq \frac{s_2}{s_1} \leq \frac{s_3}{s_2} \leq \frac{s_4}{s_3} \leq \dots,$$

implying that

$$s_n \geq s_1 s_{n-1} \geq (s_1)^2 s_{n-2} \geq \dots \geq (s_1)^n, \quad n \in \mathbb{N}.$$

On the other hand,

$$2\|x\|_T^2 = 2\langle |T|x, x \rangle \leq 2\|Tx\| \|x\| \leq \|x\|^2 + \|Tx\|^2 \stackrel{\text{def}}{=} \|x\|_{GT}^2, \quad x \in \mathcal{D}(T),$$

where the graph norm  $\|\cdot\|_{GT}$  turns  $\mathcal{D}(T)$  into a complex Hilbert space. Therefore,

$$\begin{aligned} s_n &= \|(KK^{[*]})^n x\|_T^2 \leq \frac{1}{2} \|(KK^{[*]})^n x\|_{GT}^2 \\ &\leq \frac{1}{2} \|(KK^{[*]})^n\|_{GT}^2 \|x\|_{GT}^2 \leq \frac{1}{2} \|K\|_{GT}^{2n} \|K^{[*]}\|_{GT}^{2n} \|x\|_{GT}^2. \end{aligned}$$

Consequently, for  $x \in \mathcal{D}(T)$  with  $\|x\|_T = 1$  we have

$$\|KK^{[*]}x\|_T^2 = s_1 \leq (s_n)^{1/n} \leq 2^{-1/n} \left[ \max(\|K\|, \|\tilde{K}\|) \right]^4 \|x\|_{GT}^{2/n}.$$

Letting  $n \rightarrow \infty$  and taking arbitrary  $x \in \mathcal{D}(T)$  instead of  $x \in \mathcal{D}(T)$  with  $\|x\|_T = 1$ , we obtain

$$\|KK^{[*]}x\|_T \leq \left[ \max(\|K\|, \|\tilde{K}\|) \right]^2 \|x\|_T, \quad x \in \mathcal{D}(T),$$

which in turns implies that

$$\|K^{[*]}x\|_T^2 = \langle KK^{[*]}x, x \rangle_T \leq \left[ \max(\|K\|, \|\tilde{K}\|) \right]^2 \|x\|_T^2, \quad x \in \mathcal{D}(T).$$

This proves the boundedness of  $\tilde{K} = (Q_+ - Q_-)K^{[*]}(Q_+ - Q_-)$  on  $H_T$ . To prove the boundedness of  $K$  on  $H_T$ , we repeat the above argument with  $K^{[*]}K$  instead of  $KK^{[*]}$ , using that  $\tilde{K}^*[\mathcal{D}(T)] \subset \mathcal{D}(T)$  and  $T\tilde{K}^*x = K^*Tx$  for any  $x \in \mathcal{D}(T)$ .  $\square$

Following the proof of Lemma 5.10 it is easy to show that bounded linear operators  $K$  and  $K^{[*]}$  on  $\mathcal{D}(T)$  (with respect to the graph norm) satisfying

$$\langle Kx, y \rangle_T = \langle x, K^{[*]}y \rangle_T, \quad x, y \in \mathcal{D}(T),$$

have the property that  $K$  and  $\tilde{K} \stackrel{\text{def}}{=} (Q_+ - Q_-)(K^{[*]})^*(Q_+ - Q_-)$  (where the asterisk denotes the adjoint in  $H$ ) extend to bounded linear operators on  $H_T$ . The invertibility of  $Q_+ - Q_-$  on  $H_T$  then implies that also  $(K^{[*]})^*$  extends to a bounded linear operator on  $H_T$ .

We now have the following result [24].

**Theorem 5.11.** *Let  $T$  be an injective and selfadjoint operator and  $A$  a bounded and positive selfadjoint operator with closed range, both defined on the complex Hilbert space  $H$ . Then the operator  $V$  defined by (5.26) extends to an invertible operator on  $H_T$ . Therefore, the vector-valued differential equation*

$$\psi'(x) = -T^{-1}A\psi(x), \quad 0 < x < \infty, \quad (5.29)$$

*with boundary conditions*

$$\begin{cases} \|Q_+\psi(x) - \varphi_+\|_T = 0, & x \rightarrow 0^+, \\ \|\psi(x)\|_T = O(1), & x \rightarrow +\infty, \end{cases} \quad (5.30)$$

*has a unique solution in  $H_T$  for every  $\varphi_+ \in Q_+[H_T]$ .*

*Proof.* Put

$$\begin{aligned} W &\stackrel{\text{def}}{=} 2V - I = (Q_+ - Q_-)(P_+ - P_-), \\ \tilde{W} &\stackrel{\text{def}}{=} 2\tilde{V} - I = (Q_+ - Q_-)(\tilde{P}_+ - \tilde{P}_-), \end{aligned}$$

where  $\tilde{P}_\pm = AP_\pm A^{-1}$  and  $\tilde{V} = Q_+\tilde{P}_+ + Q_-\tilde{P}_-$ . Then  $\tilde{P}_\pm$  are the spectral projections of  $TA^{-1}$  corresponding to its spectrum in  $\mathbb{R}^\pm$ . On  $\mathcal{D}(T)$  we obviously have

$$TQ_\pm = Q_\pm T, \quad TP_\pm = \tilde{P}_\pm T, \quad TV = \tilde{V}T.$$

Therefore, using that

$$W^{-1} = (P_+ - P_-)(Q_+ - Q_-), \quad \tilde{W}^{-1} = (\tilde{P}_+ - \tilde{P}_-)(Q_+ - Q_-),$$

we see from Lemma 5.10 that  $Q_\pm, P_\pm, \tilde{P}_\pm, V, \tilde{V}, W, \tilde{W}, W^{-1}$ , and  $\tilde{W}^{-1}$  extend to bounded linear operators on  $H_T$ . Thus, the extensions of  $W$  and  $\tilde{W}$  are invertible on  $H_T$ .

For  $x \in \mathcal{D}(T)$  we have

$$\begin{aligned} \langle Wx, x \rangle_T &= \langle |T|(Q_+ - Q_-)(P_+ - P_-)x, x \rangle = \langle T(P_+ - P_-)x, x \rangle \\ &= \langle A^{-1}T(P_+ - P_-)x, x \rangle_A = \langle |A^{-1}T|_A x, x \rangle_A \stackrel{\text{def}}{=} \|x\|_S^2 \geq 0, \end{aligned}$$

where  $|A^{-1}T|_A = A^{-1}T(P_+ - P_-)$  is the absolute value of  $A^{-1}T$  with respect to (5.18). Hence,

$$\|W^{1/2}x\|_T = \|x\|_S,$$

where  $W^{1/2}$  is the positive square root of  $W$  with respect to (5.28). The invertibility of  $W^{1/2}$  on  $H_T$  then implies the equivalence of the norms  $\|\cdot\|_T$  and  $\|\cdot\|_S$  on  $H_T$ .

Next, since  $W = 2V - I$ , we have

$$2\langle Vx, x \rangle_T = \|x\|_T^2 + \|x\|_S^2, \quad x \in H_T.$$

Therefore,

$$\|x\|_T^2 \leq 2\langle Vx, x \rangle_T \leq (1 + \|W^{1/2}\|_T^2)\|x\|_T^2 = (1 + \|W\|_T)\|x\|_T^2, \quad x \in H_T,$$

which implies the invertibility of  $V$  on  $H_T$ .  $\square$

We have in fact proved that the two scalar products

$$\langle x, y \rangle_T = \langle |T|x, y \rangle = \langle T(Q_+ - Q_-)x, y \rangle, \quad (5.31)$$

$$\langle x, y \rangle_S = \langle |A^{-1}T|_A x, y \rangle_A = \langle T(P_+ - P_-)x, y \rangle, \quad (5.32)$$

are equivalent on  $\mathcal{D}(T)$ . Hence their completions can be considered identical, namely the extension space  $H_T$ .

Let us now discuss the boundary value problem for  $x \in (0, \tau)$ , where  $\tau$  is finite. Assuming  $A$  strictly positive definite, the linear operator  $V_\tau$  determining the unique solvability of the problem has the form

$$V_\tau = Q_+ \left[ P_+ + e^{\tau T^{-1}A} P_- \right] + Q_- \left[ P_- + e^{-\tau T^{-1}A} P_+ \right].$$

Hence for  $x \in \mathcal{D}(T)$  we have

$$\begin{aligned} \langle V_\tau x, x \rangle_T &= \left\langle T \left[ P_+ + e^{\tau T^{-1}A} P_- \right] x, x \right\rangle - \left\langle T \left[ P_- + e^{-\tau T^{-1}A} P_+ \right] x, x \right\rangle \\ &= \left\langle |A^{-1}T|_A \left[ I - e^{-\tau |T^{-1}A|_A} \right] x, x \right\rangle_A \\ &= \left\langle \left[ I - e^{-\tau |T^{-1}A|_A} \right] x, x \right\rangle_S \geq (1 - e^{-\tau/M}) \|x\|_S^2, \end{aligned}$$

where  $\sigma(A^{-1}T) \subset [-M, M]$ . Consequently,  $V_\tau$  extends to an invertible operator on  $H_T$ . As a result, the vector-valued differential equation

$$\psi'(x) = -T^{-1}A\psi(x), \quad 0 < x < \tau, \quad (5.33)$$

with boundary conditions

$$\begin{cases} \|Q_+\psi(x) - \varphi_+\|_T = 0, & x \rightarrow 0^+, \\ \|Q_-\psi(x) - \varphi_-\|_T = 0, & x \rightarrow \tau^-, \end{cases} \quad (5.34)$$

has a unique solution in  $H_T$  for all  $\varphi_\pm \in Q_\pm[H_T]$ .

### 5.4.2 Unbounded collision operators with closed range

Let us now assume that  $T$  is injective and selfadjoint and  $A$  is positive selfadjoint with closed range on a complex Hilbert space  $H$ , while

$$H_A \stackrel{\text{def}}{=} \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T). \quad (5.35)$$

Here we drop the boundedness assumption on  $A$ . However, since  $A$  is assumed to be positive selfadjoint with closed range, the completion  $H_A$  of its domain  $\mathcal{D}(A)$  with respect to the scalar product

$$\langle x, y \rangle_A = \langle Ax, y \rangle \quad (5.36)$$

in  $H$  coincides with  $\mathcal{D}(A^{1/2})$  and differs from  $H$  if  $A$  is unbounded. We make the technical assumption (5.35), since it is satisfied in the examples to be treated in Chapter 6. It allows one to prove that  $A^{-1}T$  is a *bounded* selfadjoint operator on  $H_A$ . We let  $P_+$  and  $P_-$  stand for the orthogonal (with respect to (5.36)) projections onto the maximal  $A^{-1}T$ -invariant subspaces of  $H_A$  on which  $A^{-1}T$  is positive and negative (with respect to (5.36)), respectively. As before, by  $Q_\pm$  we denote the

orthogonal projections of  $H$  onto the maximal  $T$ -invariant subspaces on which  $T$  is positive and negative, respectively. Then  $Q_{\pm}$  leave invariant  $\mathcal{D}(T)$ .

In order to avoid introducing the completions  $H_T$  of  $\mathcal{D}(T)$  with respect to the scalar product (5.31) and  $H_S$  of  $H_A$  with respect to the scalar product (5.32) independently, we first introduce  $H_T + H_S$ . In fact, we let  $H_T + H_S$  stand for the completion of  $H_A$  with respect to the scalar product

$$\langle x, y \rangle = \inf_{\substack{x=x_1+x_2 \\ y=y_1+y_2 \\ x_1, x_2, y_1, y_2 \in H_A}} (\langle x_1, y_1 \rangle_T + \langle x_2, y_2 \rangle_S).$$

As in Subsection 5.4.1, we define  $H_T$  as the completion of  $\mathcal{D}(T)$  (within  $H_T + H_S$ ) with respect to the scalar product (5.31). Then  $T$  extends to an injective selfadjoint operator on  $H_T$ , as do the projections  $Q_{\pm}$ . We let  $H_S$  stand for the completion of  $H_A$  (within  $H_T + H_S$ ) with respect to the scalar product (5.32). Then  $A^{-1}T$  extends to a bounded selfadjoint operator on  $H_S$ , as do the projections  $P_{\pm}$ . Note that, as a result of (5.35),

$$W = (Q_+ - Q_-)(P_+ - P_-)$$

now makes sense as a linear operator from  $H_A$  into  $H_T$ .

The relations between the various spaces are generally given by the following diagram:

$$\begin{array}{ccc} \mathcal{D}(T) & \xrightarrow{\text{imbedding}} & H_T \\ \uparrow \text{imbedding} & & \\ H_A & \xrightarrow{\text{imbedding}} & H_S \end{array} \left. \vphantom{\begin{array}{ccc} \mathcal{D}(T) & \xrightarrow{\text{imbedding}} & H_T \\ \uparrow \text{imbedding} & & \\ H_A & \xrightarrow{\text{imbedding}} & H_S \end{array}} \right\} \xrightarrow{\text{imbedding}} H_T + H_S$$

**5.4.2.a General results** Let us now prove the boundedness of  $A^{-1}T$  on various spaces to enable us to list some of the spaces on which  $-T^{-1}A$  is exponentially dichotomous.

**Proposition 5.12.** *Let  $T$  be an injective and selfadjoint and  $A$  positive selfadjoint with closed range, both defined on the complex Hilbert space  $H$ , such that  $H_A \stackrel{\text{def}}{=} \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T)$ . Then the operator  $A^{-1}T$  is bounded on each of the spaces  $H_A$ ,  $\mathcal{D}(T)$  (with graph norm),  $H_T$ , and  $H_S$ . Moreover,  $-T^{-1}A$  is the generator of an analytic bisemigroup with separating projection  $P_-$  on either of the spaces  $H_A$  and  $H_S$ .*

*Proof.* Using the Closed Graph Theorem one proves with the help of (5.35) that  $TA^{-1/2} \in \mathcal{L}(H)$ . Hence,  $A^{-1/2}TA^{-1/2} \in \mathcal{L}(H)$ . Since  $A^{-1/2} : H \rightarrow H_A$  is an isometry, we conclude that  $A^{-1}T \in \mathcal{L}(H_A)$ . The injectivity and selfadjointness of  $A^{-1}T$  on  $H_A$  then imply that  $-T^{-1}A$  generates an analytic bisemigroup on  $H_A$  with separating projection  $P_- = -E(0^-; -T^{-1}A)$ .



Now observe that  $E(t; -T^{-1}A)$  (for  $0 \neq t \in \mathbb{R}$ ),  $A^{-1}T$ , and  $P_{\pm}$  commute with  $A^{-1}T$ . Then Lemma 5.10 implies that all of these operators have a bounded extension to  $H_S$ . Further,  $-T^{-1}A$  generates an analytic bisemigroup on  $H_S$ .

Next, we observe that for  $x \in \mathcal{D}(T)$  the vector  $A^{-1}Tx \in \mathcal{D}(T)$ . Now let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $\mathcal{D}(T)$  such that

$$\lim_{n \rightarrow \infty} \|x_n - x\|_{GT} = \lim_{n \rightarrow \infty} \|A^{-1}Tx_n - y\|_{GT} = 0,$$

where we recall that  $\|\cdot\|_{GT}$  is the graph norm on  $\mathcal{D}(T)$ . Then  $\|x_n - x\|$ ,  $\|Tx_n - Tx\|$ ,  $\|A^{-1}Tx_n - y\|$ , and  $\|TA^{-1}Tx_n - Ty\|$  all vanish as  $n \rightarrow \infty$ . The second and third identity imply that  $y = A^{-1}Tx$ , as a result of the boundedness of  $A^{-1}$  on  $H$ . Therefore,  $A^{-1}T$  is a closed operator on  $\mathcal{D}(T)$  and hence bounded on  $\mathcal{D}(T)$ .

Now note that for  $x, y \in \mathcal{D}(T)$ ,

$$\langle A^{-1}Tx, y \rangle_T = \langle x, (Q_+ - Q_-)A^{-1}T(Q_+ - Q_-) \rangle_T.$$

Using the remark following the proof of Lemma 5.10 it follows that  $A^{-1}T$  extends to a bounded linear operator on  $H_T$ .  $\square$

We now have the following result (essentially found in [78]).

**Theorem 5.13.** *Let  $T$  be an injective and selfadjoint operator and  $A$  a positive selfadjoint operator with closed range, both defined on the complex Hilbert space  $H$ , such that  $H_A \stackrel{\text{def}}{=} \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T)$ . Then the inverse of the operator  $V$  defined by (5.26) extends to a bounded linear operator from  $H_T$  into  $H_S$ . Therefore, the vector-valued differential equation*

$$\psi'(x) = -T^{-1}A\psi(x), \quad 0 < x < \infty, \quad (5.37)$$

*with boundary conditions*

$$\begin{cases} \|\psi(x) - V^{-1}\varphi_+\|_S = o(1), & x \rightarrow 0^+, \\ \|\psi(x)\|_S = O(1), & x \rightarrow +\infty, \end{cases} \quad (5.38)$$

*has a unique solution in  $H_S$  for every  $\varphi_+ \in Q_+[H_T]$ .*

We omit the proof. In fact, it suffices to observe that  $V = \frac{1}{2}(I + W)$  satisfies the identity

$$2\langle Vx, x \rangle_T = \|x\|_T^2 + \|x\|_S^2, \quad x \in H_A,$$

implying that

$$\|x\|_T \|x\|_S \leq \frac{1}{2}(\|x\|_T^2 + \|x\|_S^2) \leq \|Vx\|_T \|x\|_T, \quad x \in H_A,$$

so that

$$\|x\|_S \leq \|Vx\|_T, \quad x \in H_A.$$

**5.4.2.b Equivalence of the norms** Let us now derive conditions for the Hilbert spaces  $H_T$  and  $H_S$  to coincide. We begin with the following elementary result due to Beals [24, 25] (also [77, Chapter II]). Here we observe that the equivalence of (a), (b) and (c) below has not been stated in [24, 25, 77] but appears instead in [131]. Similar results have appeared in [130, 131] under the additional assumption that  $A^{-1}T$  is a compact operator on  $H_A$ .

**Theorem 5.14.** *The following statements are equivalent:*

- (a)  $H_T = H_S$ ;
- (b)  $H_T \subset H_S$ ;
- (c)  $H_S \subset H_T$ ;
- (d)  $W$  extends to a bounded linear operator from  $H_T$  into  $H_T$ ;
- (e)  $W$  extends to a bounded linear operator from  $H_S$  into  $H_T$ .

*If any of these conditions is satisfied, then the operator  $W$  extends to a bounded positive selfadjoint operator on  $H_T$  such that*

$$\|x\|_S = \|W^{1/2}x\|_T, \quad x \in H_T. \quad (5.39)$$

If the conditions (a)–(e) are satisfied, the relations between the various spaces are given by the following diagram:

$$H_A \xrightarrow{\text{imbedding}} \mathcal{D}(T) \xrightarrow{\text{imbedding}} H_T = H_S$$

*Proof.* Obviously, condition (a) implies the other four conditions. Under Condition (a) the projections  $Q_{\pm}$  and  $P_{\pm}$  extend to bounded linear operators on the coinciding spaces  $H_T$  and  $H_S$ , and hence so does  $W$ . It is then immediate that

$$\langle Wx, y \rangle_T = \langle T(P_+ - P_-)x, y \rangle = \langle A^{-1}T(P_+ - P_-)x, y \rangle_A = \langle x, y \rangle_S,$$

where  $x, y \in H_A$ , which implies (5.39).

Starting from Condition (d), we have

$$\|x\|_S^2 = \langle Wx, x \rangle_T \leq \|Wx\|_T \|x\|_T \leq \|W\|_{\mathcal{L}(H_T)} \|x\|_T^2, \quad x \in H_A,$$

which implies that  $H_T \subset H_S$ . Starting from Condition (e), we have

$$\|x\|_S^2 = \langle Wx, x \rangle_T \leq \|Wx\|_T \|x\|_T \leq \|W\|_{\mathcal{L}(H_S, H_T)} \|x\|_S \|x\|_T, \quad x \in H_A,$$

so that  $\|x\|_S \leq \|W\|_{\mathcal{L}(H_S, H_T)} \|x\|_T$  for each  $x \in H_A$ , implying that  $H_T \subset H_S$ . Thus either of (d) and (e) implies (b).

Assume Condition (c). Let  $j$  be the continuous imbedding of  $H_S$  into  $H_T$ . Then the identity

$$Wx = (Q_+ - Q_-)j(P_+ - P_-)x, \quad x \in H_S,$$

implies that  $W$  extends to a bounded operator from  $H_S$  into  $H_T$  and hence, by the above,  $H_T \subset H_S$ . As a result,  $H_T = H_S$ .

Assume condition (b). Let  $\tilde{j}$  be the continuous imbedding of  $H_T$  into  $H_S$  and let  $\tilde{W} = (P_+ - P_-)\tilde{j}(Q_+ - Q_-)$  be defined as a linear operator from  $H_A(\subset \mathcal{D}(T) \subset H_T)$  into  $H_S$ . Clearly,  $\tilde{W}$  extends to a bounded operator from  $H_T$  into  $H_S$ . Further,

$$\langle \tilde{W}x, y \rangle_S = \langle T \underbrace{\tilde{j}(Q_+ - Q_-)}_{\in \mathcal{D}(T)} x, y \rangle = \langle T(Q_+ - Q_-)x, y \rangle = \langle x, y \rangle_T,$$

where  $x, y \in H_A$ . We then estimate

$$\|x\|_T^2 \leq \|\tilde{W}\|_{\mathcal{L}(H_T, H_S)} \|x\|_T \|x\|_S, \quad x \in H_A,$$

and hence  $\|x\|_T \leq \|\tilde{W}\| \|x\|_S$  for all  $x \in H_A$ . Consequently,  $H_S \subset H_T$ , which implies that  $H_T = H_S$ .  $\square$

The following example due to Kaper et al. [101] shows that  $H_T$  and  $H_S$  need not coincide. Counterexamples involving an indefinite Sturm-Liouville problem have been given by Pyatkov [131], Abasheeva and Pyatkov [1], Fleige [63, 64], and Binding and Čurgus [29].

**Example 5.15.** Put

$$T = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \bigoplus_{n=1}^{\infty} \begin{pmatrix} n & 0 \\ 0 & n^3 \end{pmatrix},$$

on the Hilbert space  $H = \ell^2$ . Then

$$Q_{\pm} = \frac{1}{2} \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}, \quad P_{\pm} = \frac{1}{2} \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1 & \pm n \\ \pm \frac{1}{n} & 1 \end{pmatrix}.$$

We easily verify that  $T(Q_+ - Q_-)$  is the identity operator and

$$T(P_+ - P_-) = \bigoplus_{n=1}^{\infty} \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & n \end{pmatrix}.$$

Hence  $H_T$  and  $H_S$  do not coincide nor do we have  $H_T \subset H_S$  or  $H_S \subset H_T$ . The solution of the boundary value problem (5.37)–(5.38) is given by

$$e^{-xT^{-1}A}V^{-1}\varphi_+ = \left( \frac{2n}{n+1} c_n e^{-n^2 x} \begin{pmatrix} 1 \\ 1/n \end{pmatrix} \right)_{n=1}^{\infty},$$

where  $\varphi_+ = \left( c_n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)_{n=1}^{\infty}$  for complex numbers  $c_n$  with  $(c_n)_{n=1}^{\infty} \in \ell^2$ .

We now derive a sufficient condition for the equivalent conditions of Theorem 5.14 to hold true. This sufficient condition due to Beals [25] (also Čurgus [46]) regards certain operators  $X_{\pm}$  which can be constructed explicitly for kinetic equations where  $T^{-1}A$  is an indefinite Sturm-Liouville problem (see Chapter 6). Some version of this proposition, tailored towards (6.4), can be found in [22].

**Proposition 5.16.** *Let  $T$  be an injective selfadjoint operator and  $A$  a positive selfadjoint operator with closed range, both defined on a complex Hilbert space  $H$ , such that  $H_A \stackrel{\text{def}}{=} \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T)$ . Suppose there exist bounded linear operators  $X_{\pm}$  on  $H_T$  such that the following conditions are satisfied:*

- (a)  $X_{\pm}Q_{\pm} = Q_{\pm}$  and  $X_{\pm}Q_{\mp}[H_T] \subset Q_{\mp}[H_T]$ ,
- (b)  $X_{\pm}$  and  $Y_{\pm}$  leave invariant  $H_A$ , where  $(Q_+ - Q_-)Y_{\pm}$  is the adjoint of  $X_{\pm}$  in  $H_T$ .

Then  $H_T = H_S$ .

*Proof.* Put  $Z = Y_+X_+ + Y_-X_-$ . Then condition (a) implies that

$$\langle X_+y, X_+z \rangle_T = \langle X_-y, X_-z \rangle_T = 0, \quad y \in Q_+[H_T], \quad z \in Q_-[H_T]. \quad (5.40)$$

Thus for  $x \in H_T$  we have

$$\begin{aligned} \|x\|_T^2 &= \|Q_+x\|_T^2 + \|Q_-x\|_T^2 = \|X_+Q_+x\|_T^2 + \|X_-Q_-x\|_T^2 \\ &\leq \|X_+Q_+x\|_T^2 + \|X_-Q_-x\|_T^2 + \|X_+Q_-x\|_T^2 + \|X_-Q_+x\|_T^2 \\ &\stackrel{(5.40)}{=} \|X_+Q_+x\|_T^2 + \|X_-Q_-x\|_T^2 + \|X_+Q_-x\|_T^2 + \|X_-Q_+x\|_T^2 \\ &\quad + \langle X_+Q_+x, X_+Q_-x \rangle_T + \langle X_+Q_-x, X_+Q_+x \rangle_T \\ &\quad + \langle X_-Q_+x, X_-Q_-x \rangle_T + \langle X_-Q_-x, X_-Q_+x \rangle_T \\ &= \|X_+x\|_T^2 + \|X_-x\|_T^2 = \langle J(Y_+X_+ + Y_-X_-)x, x \rangle_T = [Zx, x]_T, \end{aligned}$$

where  $J = Q_+ - Q_-$ . Now put  $J_p = P_+ - P_-$ . For any  $x \in H_A$  we have

$$\begin{aligned} \|X_{\pm}e^{-t|T^{-1}A|_A}x\|_T^2 &= \langle (Q_+ - Q_-)Y_{\pm}X_{\pm}e^{-t|T^{-1}A|_A}x, e^{-t|T^{-1}A|_A}x \rangle_T \\ &= \langle A^{-1}Te^{-t|T^{-1}A|_A}x, Y_{\pm}X_{\pm}e^{-t|T^{-1}A|_A}x \rangle_A \\ &= \langle Y_{\pm}X_{\pm}e^{-t|T^{-1}A|_A}x, A^{-1}Te^{-t|T^{-1}A|_A}x \rangle_A, \end{aligned}$$

so that

$$\begin{aligned} -\frac{d}{dt}\|X_{\pm}e^{-t|T^{-1}A|_A}x\|_T^2 &= \langle X_+|T^{-1}A|_Ae^{-t|T^{-1}A|_A}x, X_+e^{-t|T^{-1}A|_A}x \rangle_T \\ &\quad + \langle X_+e^{-t|T^{-1}A|_A}x, X_+|T^{-1}A|_Ae^{-t|T^{-1}A|_A}x \rangle_T \\ &= \langle J_p e^{-t|T^{-1}A|_A}x, Y_{\pm}X_{\pm}e^{-t|T^{-1}A|_A}x \rangle_A \\ &\quad + \langle Y_{\pm}X_{\pm}e^{-t|T^{-1}A|_A}x, J_p e^{-t|T^{-1}A|_A}x \rangle_A. \end{aligned}$$

Consequently, for every  $x \in H_A$  we have

$$\begin{aligned}
& -\frac{d}{dt} \left( \|X_+ e^{-t|T^{-1}A|_A} x\|_T^2 + \|X_- e^{-t|T^{-1}A|_A} x\|_T^2 \right) \\
&= \langle J_p e^{-t|T^{-1}A|_A} x, Z e^{-t|T^{-1}A|_A} x \rangle_A + \langle Z e^{-t|T^{-1}A|_A} x, J_p e^{-t|T^{-1}A|_A} x \rangle_A \\
&\leq 2\|Z\|_{\mathcal{L}(H_A)} \left( \|e^{-t|T^{-1}A|_A} x\|_A \right)^2.
\end{aligned}$$

We now observe that  $Z$  has a restriction to  $H_A$  that is bounded on  $H_A$ . Thus for every  $x \in H_A$  we have

$$\begin{aligned}
\|x\|_T^2 &\leq \|X_+ x\|_T^2 + \|X_- x\|_T^2 \\
&= -\int_0^\infty \frac{d}{dt} \left( \|X_+ e^{-t|T^{-1}A|_A} x\|_T^2 + \|X_- e^{-t|T^{-1}A|_A} x\|_T^2 \right) dt \\
&\leq 2\|Z\|_{\mathcal{L}(H_A)} \int_0^\infty \left( \|e^{-t|T^{-1}A|_A} x\|_A \right)^2 dt \\
&= 2\|Z\|_{\mathcal{L}(H_A)} \int_0^\infty \langle e^{-2t|T^{-1}A|_A} x, x \rangle_A dt \\
&= \|Z\|_{\mathcal{L}(H_A)} \langle |A^{-1}T|_A x, x \rangle_A = \|Z\|_{\mathcal{L}(H_A)} \|x\|_S^2,
\end{aligned}$$

which shows that  $H_S \subset H_T$ . Theorem 5.14 then implies that  $H_T = H_S$ .  $\square$

From the proof of Proposition 5.16 it follows that  $Z = Y_+ X_+ + Y_- X_-$  is a bounded linear operator on  $H_T$  which is strictly positive with respect to the indefinite scalar product  $[\cdot, \cdot]$  and leaves invariant  $H_A$ . Čurgus [46] has shown the existence of such an operator  $Z$  to be equivalent to the norm equivalence  $H_T \simeq H_S$ . Of course, if  $H_T \simeq H_S$ , then  $P_+ - P_-$  is such an operator.

## Chapter 6

# Indefinite Sturm-Liouville Problems

In this chapter we apply the main results of Chapter 5 to kinetic equations which upon separation of variables reduce to Sturm-Liouville eigenvalue problems with an indefinite weight function. First second-order Sturm-Liouville problems are discussed and then higher-order problems. Various illustrative examples are given.

### 6.1 Introduction

Indefinite Sturm-Liouville problems arise from a variety of kinetic equations having the abstract form of (5.1)–(5.3), where  $A$  is a Sturm-Liouville differential operator defined on a domain of suitable functions satisfying the boundary conditions and  $T$  is the operator of multiplication by a weight function. Contrary to the case of classical Sturm-Liouville theory, the weight function changes sign, which leads to an eigenvalue equation of the form

$$(A - \lambda T)x = 0,$$

where  $-T^{-1}A$  turns out to be exponentially dichotomous (unless  $\lambda = 0$  is an eigenvalue).

The classical example is the Fokker-Planck equation for the Brownian motion of a comparatively large particle in a fluid [65, 128] given by

$$v \frac{\partial \psi}{\partial x} = \frac{\partial^2 \psi}{\partial v^2} - v \frac{\partial \psi}{\partial v}, \tag{6.1}$$

where  $v \in \mathbb{R}$  is velocity and  $x \in (0, \tau)$  (with  $0 < \tau \leq \infty$ ) is position. Writing

$\psi(x, v) = \exp(\frac{1}{4}v^2)\Phi(x, v)$  we convert (6.1) into the diffusion equation

$$v \frac{\partial \Phi}{\partial x} = \frac{\partial^2 \Phi}{\partial v^2} + \left( \frac{1}{2} - \frac{1}{4}v^2 \right) \Phi(x, v). \quad (6.2)$$

A second example regards electron scattering [35]. The corresponding kinetic equation has the form

$$\mu \frac{\partial \psi}{\partial \mu} = \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial \psi}{\partial \mu} \right), \quad (6.3)$$

where  $\mu \in (-1, 1)$  is the direction cosine and  $x \in (0, \tau)$  (with  $0 < \tau \leq \infty$ ) is position. The third example regards the, to our knowledge, first mathematically rigorous study of kinetic equations of indefinite Sturm-Liouville type [21, 22]. The equation has the form

$$x \frac{\partial \psi}{\partial t} = (-1)^{n-1} \frac{\partial^{2n} \psi}{\partial x^{2n}}, \quad (6.4)$$

where  $x \in (a, b)$  (with  $a < 0 < b$ ),  $t \in (0, \tau)$  (with  $0 < \tau \leq \infty$ ), and the  $2n$  boundary conditions  $(\partial^j \psi / \partial x^j)(a, x) = (\partial^j \psi / \partial x^j)(b, x) = 0$  ( $j = 0, 1, \dots, n-1$ ) are imposed. Our fourth example is rather artificial and has the form

$$\operatorname{sgn}(\mu) \frac{\partial \psi}{\partial x} = \frac{\partial^2 \psi}{\partial \mu^2}, \quad (6.5)$$

where  $\mu \in (-1, 1)$ ,  $x \in (0, \tau)$  (with  $0 < \tau \leq \infty$ ), and  $\psi(x, -1) = \psi(1, \mu) = 0$ . This example has the convenience that virtually all computations can be done in closed form without using special functions.

All of the equations (6.2)–(6.5) can be written in the form

$$w(v) \frac{\partial \psi}{\partial x} = -(A\psi)(x, v), \quad (6.6)$$

where  $A$  is a nonnegative selfadjoint Sturm-Liouville differential operator on  $L^2(I)$  for some subinterval  $I$  of  $\mathbb{R}$  having zero as an isolated eigenvalue of finite multiplicity and  $w(v)$  is a measurable real weight function such that  $\{v \in I : w(v) = 0\}$  has zero measure. Let  $T$  stand for the injective selfadjoint operator of multiplication by  $w$  on  $H = L^2(I)$  and put  $H_T = L^2(I; |w(v)|dv)$ . Then we seek a strongly differentiable function  $\psi : (0, \tau) \rightarrow H_T$  such that

$$\psi'(x) = -T^{-1}A\psi(x), \quad 0 < x < \tau, \quad (6.7)$$

$$\|Q_+\psi(x) - \varphi_+\|_T \rightarrow 0, \quad x \rightarrow 0^+, \quad (6.8)$$

$$\begin{cases} \|Q_-\psi(x) - \varphi_-\|_T \rightarrow 0, & x \rightarrow \tau^-, \tau \text{ finite,} \\ \|\psi(x)\|_T = O(1), & x \rightarrow \infty. \end{cases} \quad (6.9)$$

Under the condition that  $H_A \stackrel{\text{def}}{=} \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T)$  (satisfied if  $T$  is bounded, thus for the equations (6.3)–(6.5); by inspection, also for (6.1) and (6.2)), we can treat

the existence and uniqueness theory of the boundary value problems (6.7)–(6.9) as an application of the theory of Subsection 5.4.2. Since  $A^{-1}T$  is selfadjoint on  $H_A$ , there exists a resolution of the identity  $\tilde{\sigma}(\cdot)$  on  $H_A$  such that

$$A^{-1}T = \int \tau \tilde{\sigma}(d\tau). \quad (6.10)$$

The boundedness of  $A^{-1}T$  implies that there exists a compact subinterval  $[-M, M]$  of the real line such that  $\tilde{\sigma}([-M, M])$  is the identity operator on  $H_A$ . As a result,  $-T^{-1}A$  is exponentially dichotomous (and is in fact the generator of an analytic bisemigroup) on  $H_A$ . Letting  $H_S$  stand for the completion of  $H_A = \mathcal{D}(A^{1/2})$  with respect to the scalar product

$$\langle f, g \rangle_S = \langle T(\tilde{\sigma}(\mathbb{R}^+) - \tilde{\sigma}(\mathbb{R}^-))f, g \rangle, \quad (6.11)$$

we can extend the resolution of the identity  $\tilde{\sigma}(\cdot)$  from  $H_A$  to  $H_S$  and prove that  $A^{-1}T$  extends to a bounded selfadjoint operator on  $H_S$ . As a result,  $-T^{-1}A$  generates an analytic bisemigroup on  $H_S$ . According to Theorem 5.14, the crux of the existence and uniqueness theory for kinetic equations of indefinite Sturm-Liouville type is to prove that the scalar products

$$\langle f, g \rangle_T = \int_I f(\mu) \overline{g(\mu)} |w(\mu)| d\mu \quad (6.12)$$

and (6.11) are equivalent on  $H_A$  so that we can identify  $H_T$  and  $H_S$ .

The existence and uniqueness theory can be phrased in terms of certain properties of  $T^{-1}A$  as a positive selfadjoint operator with respect to the indefinite scalar product

$$[f, g]_T = \int_I f(\mu) \overline{g(\mu)} w(\mu) d\mu \quad (6.13)$$

in  $H_T = L^2(I; |w(\mu)| d\mu)$ . In fact, the necessary and sufficient condition for having a convenient existence and uniqueness theory is that  $T^{-1}A$  admits a Spectral Theorem with bounded (with respect to the norm of  $H_T$ ) resolution of the identity. For this reason we shall first discuss the Spectral Theorem for positive selfadjoint operators on a so-called Krein space and then return to the well-posedness theory of kinetic equations of indefinite Sturm-Liouville type, proving in the process that, unless zero is an eigenvalue of  $T^{-1}A$ , the operator  $-T^{-1}A$  generates an analytic bisemigroup on  $H_T = L^2(I; |w(\mu)| d\mu)$ .

When introducing the indefinite scalar product

$$[x, y]_S = \langle A^{-1}Tx, y \rangle_A = \langle Tx, y \rangle$$

on  $H_S$ , it appears that  $A^{-1}T$  is selfadjoint (and in fact positive) with respect to this scalar product.<sup>1</sup> In fact,

$$[A^{-1}Tx, x]_S = \langle A^{-1}Tx, Tx \rangle = \|A^{-1/2}Tx\|^2 \geq 0, \quad x \in H_A.$$

<sup>1</sup>Here we used the boundedness of  $A^{-1}T$  on  $H_S$ , i.e., Proposition 5.12, to prove  $A^{-1}T$  to be selfadjoint rather than to be symmetric.



The existence of the resolution of the identity  $\tilde{\sigma}$  then also follows from the Spectral Theorem of positive operators on a Krein space (cf. Subsection 6.2.1 below; also [34]), where the resolution of the identity of  $A^{-1}T$  does not have any singularities (or, in terms of Krein space theory, the critical point at zero is regular) as a result of the boundedness of  $P_{\pm} = \tilde{\sigma}(\mathbb{R}^{\pm})$  on  $H_S$ .

A special complication is the presence of an isolated eigenvalue of  $T^{-1}A$  of finite algebraic multiplicity at zero in many applications (in fact, in (6.2) and (6.3)). To modify the well-posedness theory of kinetic equations of indefinite Sturm-Liouville type to accommodate for such eigenvalues, we need to consider the indefinite scalar product

$$[x, y]_T = \langle Tx, y \rangle \quad (6.14)$$

on  $\text{Ker } A$ . The number of positive squares of (6.14) turns out to describe the nonuniqueness in solving the boundary value problem (6.7)–(6.9) for  $\tau = \infty$ .

## 6.2 Applying positive operators on Krein spaces

In this section we discuss the Spectral Theorem of positive selfadjoint operators on a Krein space and relate it to the operators studied in Section 5.4.

### 6.2.1 Positive operators on Krein spaces

Let  $H$  be a complex Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$  and let  $J$  be a signature operator on  $H$  (i.e., a selfadjoint as well as unitary operator, in other words one satisfying  $J = J^* = J^{-1}$ ). Then there exist complementary orthogonal projections  $Q_{\pm}$  on  $H$  such that  $J = Q_+ - Q_-$ . In fact,  $Q_{\pm} = \frac{1}{2}(I \pm J)$ . Now define the indefinite scalar product  $[\cdot, \cdot]$  by

$$[x, y] \stackrel{\text{def}}{=} \langle Jx, y \rangle = \langle Q_+x, Q_+y \rangle - \langle Q_-x, Q_-y \rangle, \quad x, y \in H. \quad (6.15)$$

Then  $H$  equipped with the indefinite scalar product (6.15) is called a *Krein space* with *underlying Hilbert space*  $H$ , *fundamental symmetry*  $J$ , and *fundamental decomposition*

$$H = H_+ \dot{+} H_-,$$

where  $H_{\pm} = Q_{\pm}[H]$ . If  $\dim H_- = \kappa < \infty$ , then  $H$  is called a Krein space with  $\kappa$  negative squares or a  $\Pi_{\kappa}$ -space or a *Pontryagin space*. For a detailed account of Krein spaces we refer to [108, 109, 10, 88, 34].

A vector  $x \in H$  is called *J-positive*, *J-nonnegative*, *J-negative*, *J-nonpositive*, or *J-neutral* if  $[x, x]$  is positive, nonnegative, negative, nonpositive, or zero, respectively. A linear subspace  $M$  of  $H$  is called *J-nonnegative*, *J-nonpositive*, or *J-neutral* if every  $x \in M$  is *J-nonnegative*, *J-nonpositive*, or *J-neutral*, respectively; it is called *J-positive* or *J-negative* if every nonzero vector in  $M$  is *J-positive* or

$J$ -negative, respectively. A linear subspace  $M$  of  $H$  is called *uniformly  $J$ -positive* if there exists a constant  $c > 0$  such that  $[x, x] \geq c\|x\|^2$  for each  $x \in M$ , and *uniformly  $J$ -negative* if there exists a constant  $c > 0$  such that  $[x, x] \leq -c\|x\|^2$  for each  $x \in M$ . A linear subspace  $M$  of  $H$  is called  *$J$ -indefinite* if it contains both  $J$ -positive and  $J$ -negative vectors. Two linear subspaces  $M$  and  $N$  of  $H$  are called  *$J$ -orthogonal* if  $[x, y] = 0$  for all  $x \in M$  and  $y \in N$ ;  $J$ -neutral subspaces are obviously  $J$ -orthogonal to themselves. We define the  *$J$ -orthogonal complement* of  $M$  by

$$M^{[\perp]} \stackrel{\text{def}}{=} \{x \in H : [x, y] = 0 \text{ for every } y \in M\}.$$

Let  $S \in \mathcal{L}(H)$ . Then the  *$J$ -adjoint*  $S^{[*]}$  of  $S$  is the unique bounded linear operator on  $H$  satisfying

$$[Sx, y] = [x, S^{[*]}y], \quad x, y \in H. \quad (6.16)$$

In other words,  $S^{[*]} = JS^*J$ . We call  $S \in \mathcal{L}(H)$   *$J$ -selfadjoint* if  $S^{[*]} = S$ , i.e., if  $(JS)^* = JS$ . We call  $S \in \mathcal{L}(H)$   *$J$ -positive* if  $[Sx, x] \geq 0$  for any  $x \in H$ . Obviously,  $J$ -positive operators are  $J$ -selfadjoint. Finally, a bounded linear operator  $U$  on  $H$  is called  *$J$ -unitary* if it is boundedly invertible and  $U^{-1} = U^{[*]}$  (or, in other words, if  $U$  has a bounded inverse and  $UJU^* = U^*JU = J$ ).

The following result is well known (e.g., [108, 34]).

**Proposition 6.1.** *The spectrum of a  $J$ -positive linear operator  $S \in \mathcal{L}(H)$  on a Krein space  $H$  with fundamental symmetry  $J$  is real. Moreover,*

$$\begin{cases} \text{Ker}(\lambda - S)^2 = \text{Ker}(\lambda - S), & 0 \neq \lambda \in \mathbb{R}, \\ \text{Ker} S^3 = \text{Ker} S^2. \end{cases}$$

*Proof.* Let  $0 \neq \lambda \in \mathbb{R}$  and consider vectors  $x, y \in H$  such that  $Sx = \lambda x + y$  and  $Sy = \lambda y$ . Then

$$\begin{aligned} \langle JSy, y \rangle &= [Sy, y] = \lambda[y, y] = \lambda([Sx, y] - \lambda[x, y]) \\ &= \lambda([x, Sy] - \lambda[x, y]) = \lambda^2[x, y] - \lambda^2[x, y] = 0. \end{aligned}$$

Since  $JS$  is nonnegative selfadjoint on the underlying Hilbert space, we have  $JSy = 0$  and hence  $\lambda y = 0$ . For  $0 \neq \lambda \in \mathbb{R}$  we get  $y = 0$  and hence  $\text{Ker}(S - \lambda)^2 = \text{Ker}(S - \lambda)$ . For  $\lambda = 0$  we consider vectors  $x, y, z \in H$  such that  $Sx = y$ ,  $Sy = z$ , and  $Sz = 0$ . We now proceed as follows:

$$\langle JSy, y \rangle = [Sy, y] = [z, y] = [z, Sx] = [Sz, x] = 0,$$

which implies that  $JSy = 0$ . But then  $z = J(JSy) = 0$ , which in turn implies that  $\text{Ker} S^3 = \text{Ker} S^2$ .  $\square$

We now state without proof the Spectral Theorem for  $J$ -positive operators on a Krein space [106, 109, 91, 34].

**Theorem 6.2.** *Let  $S \in \mathcal{L}(H)$  be a  $J$ -positive operator on a Krein space. Then there exists a unique operator function  $E : \mathbb{R} \setminus \{0\} \rightarrow \mathcal{L}(H)$ , the so-called spectral function of  $S$ , having the following properties:*

- 1)  $E(\lambda)$  is a  $J$ -orthogonal projection for each  $0 \neq \lambda \in \mathbb{R}$ ;
- 2)  $E(\lambda_1)E(\lambda_2) = E(\min(\lambda_1, \lambda_2))$  for  $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ ;
- 3)  $E(\lambda)[H]$  is a uniformly  $J$ -positive subspace for  $\lambda > 0$  and a uniformly  $J$ -negative subspace for  $\lambda < 0$ ;
- 4) There exist  $m, M \in \mathbb{R}$  with  $m < 0$  and  $M > 0$  such that  $E(\lambda) = 0$  for  $\lambda \leq m$  and  $E(\lambda) = I$  for  $\lambda \geq M$ ;
- 5)  $\|E(\lambda)x - E(\mu)x\| \rightarrow 0$  as  $\mu \rightarrow \lambda^-$ , for every  $x \in \mathcal{H}$ ;
- 6)  $SE(\lambda) = E(\lambda)S$  for each  $0 \neq \lambda \in \mathbb{R}$ ;
- 7) For every  $0 \neq \lambda \in \mathbb{R}$  we have

$$\sigma\left(S|_{E(\lambda)[H]}\right) \subset (-\infty, \lambda], \quad \sigma\left(S|_{(I-E(\lambda))[H]}\right) \subset [\lambda, +\infty).$$

Instead of condition 5) we also have the existence of the right strong limits:

$$5') \quad \|E(\lambda^+)x - E(\mu)x\| \rightarrow 0 \text{ as } \mu \rightarrow \lambda^+, \text{ for every } x \in \mathcal{H},$$

where  $E(\lambda) = E(\lambda^+)$  if and only if  $\lambda$  is not an eigenvalue of  $S$ . Further, the closed linear subspace  $(E(\lambda_2) - E(\lambda_1))[H]$  is  $J$ -indefinite for  $\lambda_1 < 0 < \lambda_2$ .

The spectral function allows one to represent  $S$  as follows:

$$Sx = S_0x + \int_{-\infty}^{\infty} \lambda dE(\lambda)x, \quad x \in H, \quad (6.17)$$

where  $S_0 \in \mathcal{L}(H)$  satisfies  $S_0^2 = 0$  and the integral is improper at  $\lambda = 0$ . In fact, for every  $x \in H$  we have

$$Sx = S_0x + \lim_{t \rightarrow 0^-} \int_{-\infty}^t \lambda dE(\lambda)x + \lim_{t \rightarrow 0^+} \int_t^{\infty} \lambda dE(\lambda)x,$$

where  $S_0[E(\lambda_2) - E(\lambda_1)] = 0$  for  $\lambda_1, \lambda_2 \in \mathbb{R}$  with  $\lambda_1\lambda_2 > 0$ . We now define the closed linear subspace

$$Z_0(S) = \bigcap_{\lambda_1 < 0 < \lambda_2} (E(\lambda_2) - E(\lambda_1))[H].$$

Then  $S$  and  $S_0$  leave invariant  $Z_0(S)$  and have the same restriction on  $Z_0(S)$  whose square vanishes. In general,  $Z_0(S)$  is a  $J$ -indefinite subspace of  $H$ .

We have to distinguish between two situations:

- I. For every  $x \in H$  the one-sided limits of  $E(\lambda)x$  exist, both as  $\lambda \rightarrow 0^+$  and as  $\lambda \rightarrow 0^-$ . In that case the spectral function can be extended to a spectral function  $E : \mathbb{R} \rightarrow \mathcal{L}(H)$  in a natural way and

$$Z_0(S) = (E(0^+) - E(0^-))[H].$$

In this case we call  $\lambda = 0$  a *regular critical point* of  $S$ . By the Banach-Steinhaus Theorem, the set  $\{E(\lambda) : 0 \neq \lambda \in \mathbb{R}\}$  is bounded in  $\mathcal{L}(H)$ .

- II. There exists  $x \in H$  for which at least one of the one-sided limits of  $E(\lambda)x$  as  $\lambda \rightarrow 0^+$  or as  $\lambda \rightarrow 0^-$  does not exist. In this case  $\lambda = 0$  is called an *irregular critical point* of  $S$ . As a result, the set  $\{E(\lambda) : 0 \neq \lambda \in \mathbb{R}\}$  is unbounded in  $\mathcal{L}(H)$ .

It can easily be shown that  $-S^{-1}$  is exponentially dichotomous on  $H$  if  $S$  is a bounded  $J$ -positive operator on  $H$  such that  $\text{Ker } S = \{0\}$  and  $\lambda = 0$  is a regular critical point of  $S$ . The strongly continuous bisemigroup generated by  $-S^{-1}$  is then given by

$$E(t; -S^{-1})x = \begin{cases} \int_0^\infty e^{-t/\lambda} dE(\lambda)x, & t > 0, \\ -\int_{-\infty}^0 e^{-t/\lambda} dE(\lambda)x, & t < 0. \end{cases} \quad (6.18)$$

Theorem 6.2 has been generalized to so-called definitizable operators, i.e., to operators  $S$  on a Krein space such that  $p(S)$  is  $J$ -positive for some nontrivial polynomial  $p$  [106, 109, 91, 34]. The nonreal spectrum of definitizable operators is symmetric with respect to the real line and consists of a finite number of eigenvalues of finite algebraic multiplicity.

## 6.2.2 Applications

Let  $T$  be an injective selfadjoint operator and  $A$  a positive selfadjoint operator with closed range, both defined on the complex Hilbert space  $H$ , satisfying  $H_A \stackrel{\text{def}}{=} \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T)$ . Then the completion  $H_T$  of  $\mathcal{D}(T)$  with respect to the scalar product

$$\langle x, y \rangle_T = \langle |T|x, y \rangle$$

is a Krein space with respect to the indefinite scalar product<sup>2</sup>

$$[x, y]_T = \langle Tx, y \rangle = \langle (Q_+ - Q_-)x, y \rangle_T, \quad (6.19)$$

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<sup>2</sup>Here we assume that  $T$  and  $A^{-1}T$  have both positive and negative spectrum.

where  $J = Q_+ - Q_-$  is the fundamental symmetry. Analogously, the completion  $H_S$  of  $H_A$  (which is a Hilbert space with respect to the scalar product  $\langle x, y \rangle_A = \langle A^{1/2}x, A^{1/2}y \rangle$ ) with respect to the scalar product

$$\langle x, y \rangle_S = \langle |A^{-1}T|_A x, y \rangle_A$$

is a Krein space with respect to the indefinite scalar product<sup>2</sup>

$$[x, y]_S = \langle A^{-1}Tx, y \rangle_A = \langle (P_+ - P_-)x, y \rangle_S, \quad (6.20)$$

where  $J_p = P_+ - P_-$  is the fundamental symmetry. Moreover,  $S = A^{-1}T$  is positive selfadjoint with respect to either indefinite scalar product (6.19) or (6.20).<sup>1</sup> We have shown in Subsection 5.4.2 that  $P_\pm$  are bounded on  $H_S$ , implying that  $\lambda = 0$  is a regular critical point of  $A^{-1}T$  on  $H_S$ . However, for  $\lambda = 0$  to be a regular critical point of  $A^{-1}T$  on  $H_T$  it is necessary and sufficient that  $P_\pm$  extend from  $H_A$  to bounded linear operators on  $H_T$ . This is the case if and only if  $H_T = H_S$  (see Theorem 5.14).

Let us now generalize Subsection 5.4.2. Let  $T$  be an injective selfadjoint operator and  $A$  a nonnegative selfadjoint operator with closed range, both defined on a complex Hilbert space  $H$ , such that

$$H_A \stackrel{\text{def}}{=} \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T). \quad (6.21)$$

Let  $Z_0$  be the linear subspace of  $H$  consisting of all eigenvectors and generalized eigenvectors of  $T^{-1}A$  at any zero eigenvalue. Then  $x_0 \in Z_0$  whenever there exist vectors  $x_1, \dots, x_{m-1} \in H$  such that  $x_0, \dots, x_{m-2} \in \mathcal{D}(A)$ ,  $x_1, \dots, x_{m-1} \in \mathcal{D}(T)$ ,  $Ax_k = Tx_{k+1}$  ( $k = 0, 1, \dots, m-2$ ), and  $Ax_{m-1} = 0$ . Then, by assumption,  $x_0 \in \mathcal{D}(A) \subset \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T)$  and therefore

$$Z_0 \subset \mathcal{D}(T).$$

Mimicking the proof of Proposition 6.1, it appears that

$$Z_0 = \text{Ker}(T^{-1}A)^2 \subset \mathcal{D}(T).$$

We shall **assume** the following:

- A.  $\text{Ker } A$  has finite dimension, and
- B.  $\lambda = 0$  is an isolated point of the spectrum of  $T^{-1}A$ .

Then  $Z_0$  has finite dimension. Letting  $\{\lambda \in \mathbb{C} : 0 < |\lambda| \leq \varepsilon\} \cap \sigma(T^{-1}A) = \emptyset$ , we define [149, 69]

$$P_0 = \frac{-1}{2\pi i} \oint_{|\lambda|=\varepsilon} (A - \lambda T)^{-1} T d\lambda, \quad P_0^* = \frac{-1}{2\pi i} \oint_{|\lambda|=\varepsilon} T(A - \lambda T)^{-1} d\lambda.$$

Then  $P_0$  and  $P_0^*$  are bounded projections on  $H$  such that  $Z_0 = \text{Im } P_0 \subset \mathcal{D}(T)$  and  $TP_0 = P_0^*T$ . Consequently, as a result of Lemma 5.10,  $P_0$  and  $P_0^*$  extend from  $\mathcal{D}(T)$  to bounded linear operators on  $H_T$ .

Now let  $Z_0^\dagger = \text{Im } P_0^*$ ,  $Z_1 = \text{Ker } P_0$ , and  $Z_1^\dagger = \text{Ker } P_0^*$ . Then  $Z_1$  is the orthogonal complement of  $Z_0^\dagger$  and  $Z_1^\dagger$  is the orthogonal complement of  $Z_0$  (in  $H$ ), while

$$T[Z_0] = Z_0^\dagger, \quad \overline{T[Z_1 \cap \mathcal{D}(T)]} = Z_1^\dagger, \quad (6.22)$$

$$A[Z_0] \subset Z_0^\dagger, \quad A[Z_1 \cap \mathcal{D}(A)] = Z_1^\dagger. \quad (6.23)$$

The following useful lemma can be found in [152, 77].

**Lemma 6.3.** *Let  $\beta$  be an invertible operator on  $Z_0$  such that*

$$\langle T\beta^{-1}x, x \rangle \geq 0, \quad x \in Z_0. \quad (6.24)$$

*Then*

$$A_\beta = A|_{Z_1} \dot{+} (T\beta^{-1})|_{Z_0}$$

*is a nonnegative selfadjoint operator on  $H$  with closed range and zero kernel such that  $H_{A_\beta} \stackrel{\text{def}}{=} \mathcal{D}(A_\beta^{1/2}) \subset \mathcal{D}(T)$ . Moreover,*

$$A_\beta^{-1}T = \left(T^{-1}A|_{Z_1}\right)^{-1} \dot{+} \beta. \quad (6.25)$$

Proposition 6.1 implies that the spectrum of  $\beta$  consists of a finite number of nonzero eigenvalues. Since  $Z_0 \subset \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T)$ , we have  $\mathcal{D}(A_\beta^{1/2}) = \mathcal{D}(A^{1/2})$ . We therefore define  $H_{A_\beta}$  as  $\mathcal{D}(A_\beta^{1/2}) = \mathcal{D}(A^{1/2})$  endowed with the scalar product

$$\langle x, y \rangle_{A_\beta} = \langle A_\beta x, y \rangle, \quad x, y \in \mathcal{D}(A_\beta) = \mathcal{D}(A);$$

then  $A_\beta^{-1}T$  is bounded, injective, and selfadjoint on  $H_{A_\beta}$ . Now let  $\tilde{\sigma}_\beta$  stand for the resolution of the identity of  $A_\beta^{-1}T$  and let  $P_{\pm, \beta} = \tilde{\sigma}_\beta(\mathbb{R}^\pm)$ . Then (6.25) implies that  $P_{\pm, \beta}(I - P_0)$  do not depend on  $\beta$ . Now let

$$\langle x, y \rangle_{S_\beta} = \langle |A_\beta^{-1}T|_{A_\beta} x, y \rangle_{A_\beta} = \langle T(P_{+, \beta} - P_{-, \beta})x, y \rangle, \quad x, y \in H_{A_\beta}.$$

Let  $H_{S_\beta}$  be the completion of  $H_{A_\beta}$  with respect to this scalar product. Then  $A_\beta^{-1}T$ ,  $P_{\pm, \beta}$ , and  $P_0$  extend to bounded selfadjoint operators on  $H_{S_\beta}$ , where  $P_{\pm, \beta}(I - P_0)$  do not depend on  $\beta$ . Moreover,  $A_\beta^{-1}T$  is bounded, injective, and positive selfadjoint on  $H_{S_\beta}$  with respect to the indefinite scalar product

$$[x, y]_{S_\beta} = \langle Tx, y \rangle, \quad x, y \in H_{A_\beta},$$

where  $\lambda = 0$  is a regular critical point of  $A_\beta^{-1}T$  (on  $H_{S_\beta}$ ). By contrast,  $A_\beta^{-1}T$  is bounded, injective, and positive selfadjoint on  $H_T$  with respect to the indefinite scalar product

$$[x, y]_T = \langle Tx, y \rangle, \quad x, y \in \mathcal{D}(T),$$

but the critical point  $\lambda = 0$  of  $A_\beta^{-1}T$  (on  $H_T$ ) might be irregular. It is regular if and only if  $H_T = H_{S_\beta}$ , in agreement with Theorem 5.14.

The finite-dimensional subspace  $Z_0$  of  $H_{A_\beta}$ ,  $H_T$ , or  $H_{S_\beta}$  can be equipped with the indefinite scalar product

$$[x, y] = \langle Tx, y \rangle, \quad x, y \in Z_0,$$

with respect to which  $\beta$  is positive (cf. (6.24)). Now let

$$Z_\pm \stackrel{\text{def}}{=} (P_{\pm, \beta}(I - P_0)[H_{A_\beta}] + Q_\mp[\mathcal{D}(T)]) \cap Z_0, \quad (6.26)$$

$$Z_\pm^{(T)} \stackrel{\text{def}}{=} (P_{\pm, \beta}(I - P_0)[H_{S_\beta}] + Q_\mp[H_T]) \cap Z_0. \quad (6.27)$$

Note that neither subspace of  $Z_0$  depends on  $\beta$ . The finite dimensionality of  $Z_0$  and the fact that  $H_{A_\beta}$  is dense in  $H_{S_\beta}$  and  $\mathcal{D}(T)$  is dense in  $H_T$ , imply that  $Z_\pm$  and  $Z_\pm^{(T)}$  coincide.

**Lemma 6.4.** *The subspaces  $Z_+$  and  $Z_-$  are uniformly positive and uniformly negative, respectively, and*

$$Z_+ \dot{+} Z_- = Z_0. \quad (6.28)$$

*Proof.* Let  $x_0 \in Z_0$  have the form  $x_0 = x_p + x_-$ , where  $x_p \in \text{Im } P_{+, \beta}(I - P_0)$  and  $x_- \in \text{Im } Q_-$ , where we take either definition of the image implied by (6.26). Then (6.22) implies that  $\langle Tx_0, x_p \rangle = 0$ . Therefore,

$$\underbrace{[x_p, x_p]_{S_\beta}}_{\substack{\text{nonnegative,} \\ \text{zero iff } x_p=0}} + [x_0, x_0] = \underbrace{[x_-, x_-]}_{\substack{\text{nonpositive,} \\ \text{zero iff } x_-=0}}.$$

Thus  $[x_0, x_0] \leq 0$ , while  $[x_0, x_0] = 0$  iff  $x_p = x_- = 0$ . Since  $\text{Im } P_{+, \beta}(I - P_0) \cap \text{Im } Q_- = \{0\}$ , we have  $[x_0, x_0] = 0$  iff  $x_0 = 0$ . Thus  $Z_+$  is uniformly positive. Similarly one proves that  $Z_-$  is uniformly negative.

From (6.22) and (6.23) it is easily shown that the orthogonal complement of  $Z_\pm$  in  $Z_0$  is  $Z_\mp$ . In fact,

$$\begin{aligned} (Z_\pm)^{[\perp]} &= \overline{((P_{\mp, \beta} + P_0)[H_{A_\beta}] \cap Q_\pm[\mathcal{D}(T)]) + (I - P_0)[H_{A_\beta}]} \cap Z_0 \\ &= \overline{P_{\mp, \beta}[H_{A_\beta}] \cap Q_\pm[\mathcal{D}(T)]} \cap Z_0 = Z_\mp, \end{aligned}$$

because  $Z_\pm \subset Z_0^{(T)}$  and  $Z_0$  has finite dimension.  $\square$

It is now straightforward to compute the dimensions of  $Z_\pm$ . Indeed, let decomposition  $\text{Ker } A$  be as follows:

$$\text{Ker } A = \underbrace{M_-}_{\text{negative}} \dot{+} \underbrace{M_0}_{\text{neutral}} \dot{+} \underbrace{M_+}_{\text{positive}},$$

Table 6.1: The spaces  $\text{Ker } A$  and  $Z_0$  and the dimensions of the maximal positive, negative and neutral subspaces of  $\text{Ker } A$  for various kinetic equations.

example	$m_0$	$m_+$	$m_-$	$\text{Ker } A$	$Z_0$
Eq. (5.4) ( $0 < a < 1$ )	0	0	0	$\{0\}$	$\{0\}$
Eq. (5.4) ( $a = 1$ )	1	0	0	constant functions	linear polynomials
Eq. (5.7)	1	0	0	$\text{const.} \cdot e^{-v^2/2}$	$(c_1 + c_2 v) e^{-v^2/2}$
Eq. (6.2)	1	0	0	$\text{const.} \cdot e^{-v^2/4}$	$(c_1 + c_2 v) e^{-v^2/4}$
Eq. (6.3)	1	0	0	constant functions	linear polynomials
Eq. (6.4)	0	0	0	$\{0\}$	$\{0\}$
Eq. (6.5)	0	0	0	$\{0\}$	$\{0\}$

where the three subspaces in the decomposition are orthogonal. Then the dimensions  $m_-$  of  $M_-$ ,  $m_0$  of  $M_0$ , and  $m_+$  of  $M_+$  are invariants that do not depend on the specific choice of the decomposition, while

$$\dim Z_{\pm} = m_{\pm} + m_0. \quad (6.29)$$

## 6.3 Sturm-Liouville operators of second order

In this section we discuss second-order Sturm-Liouville differential operators defined on a bounded or unbounded subinterval  $I = (a, b)$  of  $\mathbb{R}$ . First we treat the case in which the weight function is almost everywhere positive and review classical Sturm-Liouville theory. Next, we discuss the case of indefinite weight functions and prove the unique solvability of the corresponding boundary value problems (6.7)–(6.9). In the process we prove that the sufficient conditions of Proposition 5.16 are satisfied under very weak assumptions on the behavior of the indefinite weight function near its sign changes.

The creation of this chapter has primarily been inspired by the work of Beals [25] and the objective of supplying an application of exponentially dichotomous operators. This has made it imperative that any zero spectrum of  $T^{-1}A$  be an isolated eigenvalue of finite algebraic multiplicity. From this limited point of view, imposing the assumption (6.21) is natural and allows us to link this chapter to the theory expounded in Section 5.4. There also is the additional bonus of avoiding the intricate selfadjoint extension problem of symmetric linear operators on Krein spaces.

In recent years Sturm-Liouville operators  $T^{-1}A$  with an indefinite weight function (such that  $T$  is the multiplication by the weight function) have also been studied in situations in which  $A$  is selfadjoint with closed range but no longer positive [48, 26, 9]. In these situations the boundary conditions may depend on the



spectral parameter [30, 27]. One of the basic research questions has been to prove that the operator  $T^{-1}A$  is definitizable and hence allows for a Spectral Theorem. In this chapter we shall not deal with indefinite Sturm-Liouville problems of such generality.

### 6.3.1 Positive weight functions

Following [162], by a *Sturm-Liouville differential expression of the second order* we mean

$$(\tau f)(\mu) = \frac{1}{w(\mu)} \left\{ - (p(\mu)f'(\mu))' + q(\mu)f(\mu) \right\} \quad \text{in } (a, b), \quad (6.30)$$

where  $I = (a, b)$  is an arbitrary bounded or unbounded interval. We make the following assumptions:<sup>3</sup>

- (a)  $p, q$ , and  $w$  are real measurable functions such that  $\{1/p, q, w\} \subset L^1_{\text{loc}}(a, b)$ , i.e., such that  $1/p, q$ , and  $w$  are integrable on each compact subinterval of  $(a, b)$ ,
- (b)  $p(\mu) > 0$  and  $w(\mu) > 0$  for almost every  $\mu \in (a, b)$ .

Many proofs in [162] run more smoothly if we replace (a) and (b) by the following stronger assumptions:<sup>2</sup>

- (a')  $q$  and  $w$  are piecewise continuous real functions on  $(a, b)$ , while  $p$  is continuous and piecewise  $C^1$ ,
- (b') we have  $p(\mu) > 0$  and  $w(\mu) > 0$  for all  $\mu \in (a, b)$ .

Let  $g : (a, b) \rightarrow \mathbb{C}$  be measurable and let  $\lambda \in \mathbb{C}$ . Then a function  $f : (a, b) \rightarrow \mathbb{C}$  is called a *solution* of the equation

$$(\tau - \lambda)f = g$$

if  $f$  and  $pf'$  are absolutely continuous<sup>4</sup> and

$$-(pf')'(\mu) + (q(\mu) - \lambda w(\mu))f(\mu) = w(\mu)g(\mu)$$

for almost every  $\mu \in (a, b)$ .

On the Hilbert space  $L^2(I; w d\mu)$  we define the *maximal operator*  $A_M$  by

$$\begin{cases} \mathcal{D}(A_M) = \left\{ f \in L^2(I; w d\mu) : \begin{array}{l} f, pf' \text{ are absolutely continuous in } I \\ \tau f \in L^2(I; w d\mu) \end{array} \right\}, \\ A_M f = \tau f. \end{cases}$$

We define  $A'_m$  as the restriction of  $A_M$  to

$$\mathcal{D}(A'_m) = \{f \in \mathcal{D}(A_M) : f \text{ has compact support in } I\}.$$

<sup>3</sup>We deviate from [162] by taking  $p$  positive instead of either positive or negative.

<sup>4</sup>Under the assumptions (a') and (b'), the function  $pf'$  is absolutely continuous if and only if  $f'$  is absolutely continuous.

Then  $A'_m$  is a symmetric operator on  $L^2(I; w d\mu)$  having at least one selfadjoint extension. Thus we can define the *minimal operator*

$$A_m = \overline{A'_m}$$

as the minimal closed extension of  $A'_m$ . Given  $f, g : (a, b) \rightarrow \mathbb{C}$  such that  $pf', pg'$  are absolutely continuous, we have for  $[\alpha, \beta] \subset (a, b)$  the *Lagrange identity*

$$\int_{\alpha}^{\beta} \left\{ \overline{(\tau f)(\mu)} g(\mu) - \overline{f(\mu)} (\tau g)(\mu) \right\} w(\mu) d\mu = [f, g]_{\beta} - [f, g]_{\alpha} \stackrel{\text{def}}{=} [f, g]_{\alpha}^{\beta},$$

where  $[f, g]_{\mu} \stackrel{\text{def}}{=} \overline{f(\mu)} (pg')(\mu) - \overline{(pf')(\mu)} g(\mu)$  for  $\mu \in (a, b)$ . Thus for  $f, g \in \mathcal{D}(A_M)$  the limits of  $[f, g]_{\mu}$  as  $\mu \rightarrow a^+$  and  $\mu \rightarrow b^-$  exist and are finite, while in  $L^2(I; w d\mu)$  we have

$$\langle A_M f, g \rangle - \langle f, A_M g \rangle = [f, g]_a^b.$$

An extension  $A$  of  $A'_m$  (or of  $A_m$ ) is selfadjoint if and only if

$$\mathcal{D}(A) = \{g \in \mathcal{D}(A_M) : [f, g]_a^b = 0 \text{ for each } f \in \mathcal{D}(A)\}.$$

The Sturm-Liouville differential expression (6.30) is called *regular at  $a$*  if  $a > -\infty$  and  $\{1/p, q, w\} \subset L^1(a, c)$  for any  $c \in (a, b)$ , and *singular at  $a$*  if it is not regular at  $a$ . The expression (6.30) is called *regular at  $b$*  if  $b < +\infty$  and  $\{1/p, q, w\} \subset L^1(c, b)$  for any  $c \in (a, b)$ , and *singular at  $b$*  if it is not regular at  $b$ . The expression (6.30) is called *regular* if it is regular at  $a$  and at  $b$ , and *singular* otherwise. Regular differential expressions have the property that, for each  $f \in \mathcal{D}(A_M)$ , the functions  $f$  and  $pf'$  have finite limits as  $\mu \rightarrow a^+$  and  $\mu \rightarrow b^-$ . For regular differential expressions of the form (6.30) the selfadjoint extensions  $A$  of  $A_m$  are all bounded below and are characterized by pairs of  $2 \times 2$  matrices  $B_a$  and  $B_b$  satisfying

$$\left\{ \begin{array}{l} \begin{pmatrix} B_a & B_b \end{pmatrix} \text{ has rank } 2, \\ B_a J B_a^* = B_b J B_b^*, \text{ where } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{array} \right.$$

In this case we have

$$\mathcal{D}(A) = \left\{ f \in \mathcal{D}(A_M) : B_a \begin{pmatrix} f(a) \\ (pf')(a) \end{pmatrix} = B_b \begin{pmatrix} f(b) \\ (pf')(b) \end{pmatrix} \right\}.$$

Further,  $A$  has a pure eigenvalue spectrum (with eigenvalues of multiplicity  $\leq 2$ ), where the eigenvalues  $\lambda_n$  satisfy  $\sum_n |\lambda_n|^{-2} < \infty$ . The selfadjoint extensions  $A_{\alpha, \beta}$  (with  $\alpha, \beta \in [0, \pi)$ ) of  $A_m$  with separated boundary conditions are given by

$$\mathcal{D}(A_{\alpha, \beta}) = \left\{ f \in \mathcal{D}(A_M) : \begin{array}{l} f(a) \cos \alpha - (pf')(a) \sin \alpha = 0 \\ f(b) \cos \beta - (pf')(b) \sin \beta = 0 \end{array} \right\}.$$

Under separated boundary conditions the eigenvalues of  $A$  are all simple.

Now choose  $c \in (a, b)$ . Then the differential expression (6.30) has exactly one of the following two properties [the so-called *Weyl alternative* at  $a$ ]:

- a<sub>1</sub>. For every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists exactly one linearly independent solution  $u$  of  $(\tau - \lambda)u = 0$  such that  $u|_{(a,c)} \in L^2((a, c); w d\mu)$ . Then  $\tau$  is said to be a *limit point* at  $a$ . We then have  $[f, g]_a = 0$  for all  $f, g \in \mathcal{D}(A_M)$ .
- b<sub>1</sub>. For every  $\lambda \in \mathbb{C}$  all solutions  $u$  of  $(\tau - \lambda)u = 0$  have the property that  $u|_{(a,c)} \in L^2((a, c); w d\mu)$ . Then  $\tau$  is said to be a *limit circle* at  $a$ .

Similarly, the differential expression (6.30) also has one of the following two properties [the so-called *Weyl alternative* at  $b$ ]:

- a<sub>2</sub>. For every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists exactly one linearly independent solution  $u$  of  $(\tau - \lambda)u = 0$  such that  $u|_{(c,b)} \in L^2((c, b); w d\mu)$ . Then  $\tau$  is said to be a *limit point* at  $b$ . We then have  $[f, g]_b = 0$  for all  $f, g \in \mathcal{D}(A_M)$ .
- b<sub>2</sub>. For every  $\lambda \in \mathbb{C}$  all solutions  $u$  of  $(\tau - \lambda)u = 0$  have the property that  $u|_{(c,b)} \in L^2((c, b); w d\mu)$ . Then  $\tau$  is said to be a *limit circle* at  $b$ .

If  $\tau$  is a limit point at both endpoints, then the maximal operator  $A_M$  is selfadjoint and no boundary conditions need to be imposed to single out the domain of selfadjointness. If  $\tau$  is a limit point at one endpoint and a limit circle at the other, then a boundary condition at the latter endpoint suffices to select a domain of selfadjointness. Finally, if  $\tau$  is a limit circle at both endpoints, two boundary conditions are required to define a domain of selfadjointness. We refer to Theorems 13.20 and 13.21 of [162] for details.

### 6.3.2 Indefinite weight functions

Given a real weight function  $w \in L^1_{\text{loc}}(I)$  defined on an arbitrary bounded or unbounded interval  $I = (a, b)$  which has at most finitely many zeros, we consider the differential expression

$$(\tau f)(\mu) = -(p(\mu)f'(\mu))' + q(\mu)f(\mu) \quad \text{in } (a, b), \quad (6.31)$$

which is obtained from (6.30) by putting  $w(\mu) \equiv 1$  and, apart from having more leeway as to the sign of  $w(\mu)$ , satisfies conditions (a) and (b) [or, alternatively, (a') and (b')]. Let  $A$  be a nonnegative selfadjoint extension of the corresponding minimal operator  $A_m$  on  $H = L^2(I; d\mu)$  and suppose  $A$  has a closed range. Let us define the injective selfadjoint operator  $T$  on  $H$  as follows:

$$\begin{cases} \mathcal{D}(T) = \{f \in L^2(I; d\mu) : wf \in L^2(I; d\mu)\}, \\ (Tf)(\mu) = w(\mu)f(\mu). \end{cases}$$

Then the completion  $H_T$  of  $\mathcal{D}(T)$  with respect to the scalar product  $\langle f, g \rangle_T = \langle Tf, g \rangle$  ( $f, g \in \mathcal{D}(T)$ ) is given by  $H_T = L^2(I; |w| d\mu)$ .

Now observe that the differential expression

$$(\tau f)(\mu) = \frac{1}{|w(\mu)|} \left\{ - (p(\mu)f'(\mu))' + q(\mu)f(\mu) \right\} \quad \text{in } (a, b) \quad (6.32)$$

satisfies the conditions of Subsection 6.3.1. By suitably extending the corresponding minimal operator we obtain its selfadjoint extensions on  $H_T = L^2(I; |w|d\mu)$ , one of which coincides with  $|T|^{-1}A$ . We call the differential expression (6.30) with indefinite weight  $w$  having at most finitely many zeros in  $I$  regular (at  $a$ , at  $b$ , or just regular), singular (at  $a$ , at  $b$ , or just singular), a limit point at  $a$  or  $b$ , or a limit circle at  $a$  and  $b$  if the differential expression (6.32) has the same property. These definitions make perfect sense, because we have avoided sign changes of the weight function converging to either endpoint of  $I$ . We shall assume that  $T^{-1}A$  (or, equivalently,  $|T|^{-1}A$ ) has a closed range in  $H_T$ . As a result, if  $\lambda = 0$  is an eigenvalue of  $T^{-1}A$ , it is necessarily isolated and has finite algebraic multiplicity. The generalized eigenvector space  $Z_0$  at any zero eigenvalue satisfies

$$Z_0 = \text{Ker}(T^{-1}A)^2 \subset \mathcal{D}(T).$$

We shall make the technical assumption (also made in [25]) that

$$H_A \stackrel{\text{def}}{=} \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T). \quad (6.33)$$

This assumption is obviously satisfied if the weight function  $w$  is bounded, but it is also satisfied in the case of the example in (6.2).

Let us now present three instructive examples where (6.33) holds true.

**Example 6.5.** To solve (6.5), consider  $I = (-1, 1)$ ,  $w(\mu) = \text{sgn}(\mu)$ , and  $(Af)(\mu) = -f''(\mu)$  with Dirichlet boundary conditions, so that  $H_T = H = L^2(I)$ . Then any positive eigenvalue  $\lambda$  of  $T^{-1}A$  has an eigenfunction of the form

$$\psi(\mu, \lambda) = \begin{cases} \sinh(\sqrt{\lambda}) \sin(\sqrt{\lambda}(1 - \mu)), & 0 \leq \mu \leq 1, \\ \sin(\sqrt{\lambda}) \sinh(\sqrt{\lambda}(1 + \mu)), & -1 \leq \mu \leq 0, \end{cases}$$

where  $A(\lambda) \stackrel{\text{def}}{=} \tanh(\sqrt{\lambda}) = -\tan(\sqrt{\lambda})$ . Letting  $\lambda_n$  ( $n = 1, 2, 3, \dots$ ) stand for the  $n$ -th positive zero of the equation  $\tanh(\sqrt{\lambda}) = -\tan(\sqrt{\lambda})$  [see Figure 6.1], we have the corresponding eigenfunctions

$$\phi_n(\mu) = \begin{cases} \sqrt{1 + A(\lambda_n)^2} \sin(\sqrt{\lambda_n}(1 - \mu)), & 0 \leq \mu \leq 1, \\ (-1)^n \sqrt{1 - A(\lambda_n)^2} \sinh(\sqrt{\lambda_n}(1 + \mu)), & -1 \leq \mu \leq 0, \end{cases}$$

where  $[T\phi_n, \phi_n] = \int_{-1}^1 \text{sgn}(\mu) |\phi_n(\mu)|^2 d\mu = 1$ . The negative eigenvalues are given by  $\lambda_{-n} = -\lambda_n$ , with corresponding normalized eigenfunctions  $\phi_{-n}(\mu) = \phi_n(-\mu)$ . The well-posedness of the corresponding diffusion problem (6.5) was first established by different means in [126].

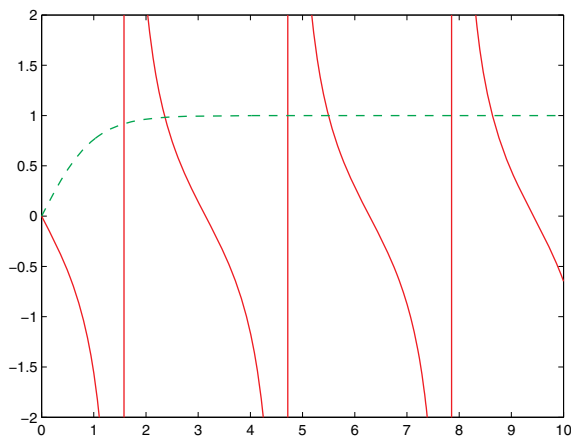


Figure 6.1: The plot contains the graphs of  $\tanh(x)$  and  $-\tan(x)$  with intersection points  $\sqrt{\lambda_n}$ .

**Example 6.6.** To solve (6.2), consider  $I = \mathbb{R}$ ,  $w(v) = v$ , and  $(Af)(v) = -f''(v) - (\frac{1}{2} - \frac{1}{4}v^2)f(v)$ , where the differential expression  $(\tau f)(v) = -f''(v) - (\frac{1}{2} - \frac{1}{4}v^2)f(v)$  is a limit point at both endpoints  $\pm\infty$  of  $I = \mathbb{R}$ . We have  $H = L^2(\mathbb{R})$  and  $H_T = L^2(\mathbb{R}; |v|dv)$ . There exists a unique vector  $\psi_0 \in H$  (up to a constant factor) satisfying  $A\psi_0 = T\psi_0$  and  $\langle T\psi_0, \psi_0 \rangle = 0$ , namely  $\psi_0(v) = v \exp(-\frac{1}{4}v^2)$ . The nonzero eigenvalues of  $T^{-1}A$  are given by  $\lambda_{\pm n} = \pm\sqrt{n}$  with corresponding eigenfunctions (cf. Figure 6.2)

$$\varphi_{\pm n}(v) = \pm \frac{1}{\sqrt{(2\pi)^{1/2} n^{3/2} 2^{n+1} (n!)}} e^{-\frac{1}{4}v^2} e^{\pm v\sqrt{n}} H_n\left(\frac{v \mp 2\sqrt{n}}{\sqrt{2}}\right),$$

where  $\langle T\varphi_{\pm n}, \varphi_{\pm n} \rangle = \pm 1$  and  $n = 1, 2, 3, \dots$ . Here

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}$$

is the Hermite polynomial of degree  $n$  (cf. [2, 22.6.21 and 22.5.18]).

**Example 6.7.** To solve (6.3), consider  $I = (-1, 1)$ ,  $w(\mu) = \mu$ , and

$$(Af)(\mu) = -((1 - \mu^2)f')'$$

defined on a suitable domain of functions having finite limits at both endpoints. Then there exists a unique function (up to a constant factor)  $\phi_0(\mu) = 1$  satisfying  $A\phi_0 = 0$ , while  $\langle T\phi_0, \phi_0 \rangle = 0$ . The operator  $T^{-1}A$  has a pure eigenvalue spectrum  $\{\lambda_n\}_{n=-\infty}^{\infty}$ , where  $\lambda_0 = 0$  and  $\lambda_{-n} = -\lambda_n$ . Expanding the eigenfunctions with

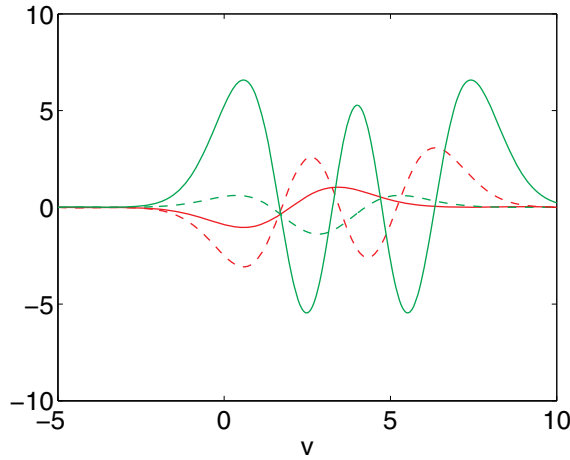


Figure 6.2: The plot contains the graphs of the eigenfunctions  $\varphi_n(v)$  for  $n = 1, 2, 3, 4$ . Note that  $\phi_n(x)$  has  $n$  zeros, all of them positive.

respect to the  $L^2$ -normalized Legendre polynomials  $p_j = \sqrt{j + \frac{1}{2}} P_j$ ,

$$\phi(\mu) = \sum_{j=0}^{\infty} c_j(\lambda) p_j(\mu) = \sum_{j=0}^{\infty} c_j(\lambda) \sqrt{j + \frac{1}{2}} P_j(\mu),$$

we obtain the semi-infinite linear system

$$j(j+1)c_j(\lambda) = \lambda \{ \alpha_{j-1}c_{j-1}(\lambda) + \alpha_j c_{j+1}(\lambda) \}, \quad j = 0, 1, 2, \dots,$$

where  $\alpha_{-1} = 0$  and  $\alpha_j = (j+1)/\sqrt{(2j+1)(2j+3)}$ . For  $\lambda \neq 0$  this leads to  $c_1 = 0$ ,  $c_0 = -(2/\sqrt{5})c_2$ , and

$$\lambda^{-1}d_j(\lambda) = \beta_{j-1}d_{j-1}(\lambda) + \beta_j d_{j+1}(\lambda), \quad j = 2, 3, 4, \dots,$$

where  $d_j(\lambda) = \sqrt{j(j+1)}c_j(\lambda)$  and  $\beta_j = \alpha_j/((j+1)\sqrt{j(j+2)})$ . The nonzero eigenvalues are therefore the reciprocals of the eigenvalues of the Jacobi matrix  $\mathbb{T}$  given by

$$\mathbb{T}_{j,l} = \begin{cases} \beta_{\min(j,l)+2}, & |j-l| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $j, l = 0, 1, \dots$ . Letting  $\mathbb{T}^{(J)}$  stand for its left upper  $J \times J$  corner, then  $p_0(\zeta) = 1$  and  $p_J(\zeta) = \det(I_J - \mathbb{T}^{(J)})$  form an orthogonal system of polynomials with respect to the (even) probability measure  $\nu$  of compact support [55, Theorem 2.31] given by

$$\langle (\zeta - \mathbb{T})^{-1}e_0, e_0 \rangle_{\ell^2} = \int \frac{d\nu(\tau)}{\zeta - \tau} = \sum_{0 \neq j \in \mathbb{Z}} \frac{|\langle e_0, \phi_j \rangle|^2}{\zeta - (1/\lambda_j)},$$

where  $(e_0)_j = \delta_{0,j}$  and  $\{\phi_j\}_{j \neq 0}$  is an orthonormal basis of eigenvectors of  $\mathbb{T}$ . The eigenvalues and corresponding eigenfunctions are not known in closed form. Their asymptotics as  $n \rightarrow \pm\infty$  have been worked out in [161].

To prove the well-posedness of the boundary value problems, we derive the following result due to Beals [25]. In the second-order case an alternative proposition leading to the norm equivalence  $H_T \simeq H_S$  appeared in [61].

**Lemma 6.8.** *Let  $a < \tau_1 < \dots < \tau_m < b$  be the sign changes of the weight function and suppose  $0 < \varepsilon < \frac{1}{2} \min(\tau_1 - a, \tau_2 - \tau_1, \dots, b - \tau_m)$ . Assume that for  $\alpha_1, \dots, \alpha_m > -\frac{1}{2}$  and  $m_j \in C^1(\tau_j - \varepsilon, \tau_j + \varepsilon)$  with  $m_j(\tau_j) \neq 0$  ( $j = 1, \dots, m$ ) we have*

$$w(\mu) = (-1)^{m-j} \operatorname{sgn}(\mu - \tau_j) |\mu - \tau_j|^{\alpha_j} m_j(\mu).$$

*Then there exist linear operators  $X_{\pm}$  and  $Y_{\pm}$  satisfying the conditions of Proposition 5.16.*

*Proof.* Suppose  $I = (a, b)$  with  $a < 0 < b$ ,  $w(\mu) > 0$  for  $\mu \in (0, b)$ , and  $w(\mu) < 0$  for  $\mu \in (a, 0)$ . Let  $w(\mu) = \operatorname{sgn}(\mu) |\mu|^{\alpha} m(\mu)$  for  $\mu \in (-\varepsilon, \varepsilon) \subset (a, b)$ ,  $\alpha > -\frac{1}{2}$ ,  $m(0) \neq 0$ , and  $m \in C^1(-\varepsilon, \varepsilon)$ . Take  $t_1$  and  $t_2$  with  $0 < t_1 < t_2 \leq 1$  and  $\varphi \in C^\infty(\mathbb{R})$  real-valued satisfying  $\varphi(0) = 1$  and support contained in  $(t_1 a, t_1 b)$ , and let us choose  $t_1$  and  $t_2$  in such a way that  $t_1 \varepsilon$  and  $-t_1 \varepsilon$  do not belong to the complex hull of the support of  $\varphi$ . Put

$$g_j(\mu) = -\frac{w(\mu)}{w(-t_j \mu)} = (t_j)^{-\alpha} \frac{m(\mu)}{m(-t_j \mu)}, \quad \mu \in (-\varepsilon, \varepsilon), \quad j = 1, 2.$$

Then  $\varphi g_1$  and  $\varphi g_2$  are  $C^1$  in an open interval containing the support of  $\varphi$ . For certain constants  $c_1$  and  $c_2$  to be selected below we now define

$$(X_+ f)(\mu) = \begin{cases} f(\mu), & 0 < \mu < b, \\ \varphi(\mu) \{c_1 t_1 f(-t_1 \mu) + c_2 t_2 f(-t_2 \mu)\}, & a < \mu < 0. \end{cases}$$

In order that  $(Q_+ - Q_-)Y_+$  is the  $H_T$ -adjoint of  $X_+$ , we define

$$(Y_+ f)(\mu) = \begin{cases} f(\mu) + c_1(\varphi g_1 f)(-\mu/t_1) + c_2(\varphi g_2 f)(-\mu/t_2), & 0 < \mu < b, \\ 0, & a < \mu < 0. \end{cases}$$

Then  $X_+ Q_+ = Q_+$ ,  $X_+ Q_- = 0$ , and  $X_+$  and  $Y_+$  are bounded on  $H_T$ .

For  $X_+$  to map  $\mathcal{D}(A^{1/2})$  into itself, it is necessary and sufficient that it does not create a jump discontinuity at  $\mu = 0$ , i.e., that

$$c_1 t_1 + c_2 t_2 = 1. \tag{6.34}$$

Further, if  $Y_+$  is to map  $\mathcal{D}(A^{1/2})$  into  $\mathcal{D}(A^{1/2})$ , we must require

$$c_1 g_1(0^-) + c_2 g_2(0^-) = -1. \tag{6.35}$$

Since the determinant of the linear system (6.34)–(6.35), namely

$$t_1 g_2(0^-) - t_2 g_1(0^-) = t_1 t_2 (t_2^{-1-\alpha} - t_1^{-1-\alpha}),$$

is nonzero, we can solve the linear system (6.34)–(6.35) uniquely for  $c_1$  and  $c_2$ . Hence  $X_+$  and  $Y_+$  satisfy the conditions of Proposition 5.16. In the same way we construct  $X_-$  and  $Y_-$ .

The general case proceeds by using a  $C^\infty$  partition of unity. Let  $\{U_j\}_{j \in J}$  be an open cover of  $I$  such that each  $U_j$  contains at most one sign change of  $w$ . Let  $\{\varphi_j\}_{j \in J}$  be a  $C^\infty$  partition of unity subordinated to the cover  $\{U_j\}_{j \in J}$  (cf. [104, 132]). This means that each  $\varphi_j$  is a nonnegative  $C^\infty$  function on  $I$  with support contained in  $U_j$  such that for every  $\mu \in I$  we have  $\varphi_j(\mu) \neq 0$  for at most finitely many  $j \in J$ , while  $\sum_{j \in J} \varphi_j(\mu) = 1$  for  $\mu \in I$ . For every  $j \in J$  we construct the operators  $X_{\pm,j}$  and  $Y_{\pm,j}$  of the above type as if the weight function  $w$  only changes sign in  $U_j$ . We then define

$$\begin{aligned} (X_\pm f)(\mu) &= \sum_{j \in J} \varphi_j(\mu) (X_{\pm,j} f)(\mu), \\ (Y_\pm f)(\mu) &= \sum_{j \in J} \varphi_j(\mu) (Y_{\pm,j} f)(\mu). \end{aligned}$$

Then  $X_\pm$  and  $Y_\pm$  satisfy the conditions of the lemma and therefore those of Proposition 5.16.  $\square$

Since the hypotheses of Proposition 5.16 are fulfilled, we now have

$$H_T = H_{S_\beta}. \quad (6.36)$$

As a consequence of (6.36) and Theorem 5.13, we obtain

**Theorem 6.9.** *For every  $\varphi_+ \in Q_+[H_T]$  the vector-valued differential equation*

$$\psi'(x) = -T^{-1}A\psi(x), \quad 0 < x < \infty,$$

*with boundary conditions*

$$\begin{cases} \|Q_+\psi(x) - \varphi_+\|_T = o(1), & x \rightarrow 0^+, \\ \|\psi(x)\|_T = O(1), & x \rightarrow +\infty, \end{cases}$$

*has at least one solution, which is unique whenever  $(T\phi, \phi) \geq 0$  for every  $\phi \in \text{Ker } A$ .*

## 6.4 Sturm-Liouville operators of any order

We now generalize Section 6.3 to higher-order Sturm-Liouville differential operators defined on a bounded or unbounded subinterval  $I = (a, b)$  of  $\mathbb{R}$ . Although we treat the subject matter with less detail, we again distinguish between the cases of positive and indefinite weight functions.



### 6.4.1 Positive weight functions

Following [124], by a *Sturm-Liouville differential expression of the 2nth order* we mean

$$(\tau f)(\mu) = \frac{1}{w(\mu)} \sum_{j=0}^n (-1)^j \left( p_{n-j}(\mu) f^{(j)}(\mu) \right)^{(j)}, \quad (6.37)$$

where  $I = (a, b)$  is an arbitrary bounded or unbounded interval. We make the following assumptions:

(a)  $p_0, p_1, \dots, p_n$  are real measurable functions such that

$$\{1/p_0, p_1, \dots, p_n, w\} \subset L^1_{\text{loc}}(a, b),$$

i.e., such that  $1/p_0, p_1, \dots, p_n, w$  are integrable on each compact subinterval of  $(a, b)$ ,

(b)  $p_0(\mu) > 0$  and  $w(\mu) > 0$  for almost every  $\mu \in (a, b)$ .

Many proofs in [124] run more smoothly if we replace (a) and (b) by the following stronger assumptions:<sup>5</sup>

(a') for  $j = 0, 1, \dots, n$  we have  $p_j \in C^{n-j}(I)$  and real,

(b') we have  $p_0(\mu) > 0$  and  $w(\mu) > 0$  for all  $\mu \in (a, b)$ .

We now define the *quasi-derivatives*  $y^{[k]}$  ( $k = 0, 1, \dots, 2n$ ) as follows:

$$\begin{cases} y^{[0]} = y, \\ y^{[j]} = \frac{d^j y}{dx^j}, & j = 1, \dots, n-1, \\ y^{[n]} = p_0 \frac{d^n y}{dx^n}, \\ y^{[n+j]} = p_j \frac{d^{n-j} y}{dx^{n-j}} - \frac{d}{dx} (y^{[n+j-1]}), & j = 1, \dots, n. \end{cases}$$

Equation (6.37) can then be written in the form

$$(\tau f)(\mu) = \frac{1}{w(\mu)} f^{[2n]}(\mu). \quad (6.38)$$

Letting  $\lambda \in \mathbb{C}$  and letting  $g : (a, b) \rightarrow \mathbb{C}$  be measurable, a function  $f : (a, b) \rightarrow \mathbb{C}$  is called a *solution* of the equation

$$(\tau - \lambda)f = g$$

---

<sup>5</sup>For  $n = 1$  we impose assumptions that are somewhat stronger than those given in Section 6.3 without restricting the theory in an essential way. Contrary to [124] we fix the sign of  $p_0(\mu)$  and let the weight be part of the definition of a differential expression.

if and only if all quasi-derivatives of  $f$  up to the  $(2n - 1)$ th order exist and are absolutely continuous in each compact subinterval of  $(a, b)$  and

$$\sum_{j=0}^n (-1)^j \left( p_{n-j}(\mu) f^{(j)}(\mu) \right)^{(j)} - \lambda f(\mu) = w(\mu) f(\mu)$$

for every  $\mu \in (a, b)$ .

On the Hilbert space  $L^2(I; w d\mu)$  we define the *maximal operator*  $A_M$  on the (dense) domain

$$\mathcal{D}(A_M) = \left\{ f \in L^2(I; w d\mu) : \begin{array}{l} \text{all quasi-derivatives of } f \text{ up to the} \\ (2n - 1)\text{th order exist and are} \\ \text{absolutely continuous in each} \\ \text{compact subinterval of } (a, b), \\ \tau f \in L^2(I; w d\mu) \end{array} \right\}$$

by  $A_M f = \tau f$ . We define  $A'_m$  as the restriction of  $A_M$  to

$$\mathcal{D}(A'_m) = \{f \in \mathcal{D}(A_M) : f \text{ has compact support in } I\}.$$

Then  $A'_m$  is a symmetric operator on  $L^2(I; w d\mu)$  having at least one selfadjoint extension. We then define the *minimal operator*

$$A_m = \overline{A'_m}$$

as the minimal closed extension of  $A'_m$ . Given  $f, g : (a, b) \rightarrow \mathbb{C}$  such that all of their quasi-derivatives up to the  $(2n - 1)$ th order exist and are absolutely continuous on each compact subinterval  $[\alpha, \beta]$  of  $(a, b)$ , we have the *Lagrange identity*

$$\int_{\alpha}^{\beta} \left\{ \overline{(\tau f)(\mu)} g(\mu) - \overline{f(\mu)} (\tau g)(\mu) \right\} w(\mu) d\mu = [f, g]_{\beta} - [f, g]_{\alpha} \stackrel{\text{def}}{=} [f, g]_{\alpha}^{\beta},$$

where  $[f, g]_{\mu} \stackrel{\text{def}}{=} \sum_{j=1}^n \{ f^{[j-1]} \bar{g}^{[2n-j]} - f^{[2n-j]} \bar{g}^{[j-1]} \}$ . Thus for  $f, g \in \mathcal{D}(A_M)$  the limits of  $[f, g]_{\mu}$  as  $\mu \rightarrow a^+$  and  $\mu \rightarrow b^-$  exist and are finite, while in  $L^2(I; w d\mu)$  we have

$$\langle A_M f, g \rangle - \langle f, A_M g \rangle = [f, g]_a^b.$$

An extension  $A$  of  $A'_m$  (or of  $A_m$ ) is selfadjoint if and only if

$$\mathcal{D}(A) = \{g \in \mathcal{D}(A_M) : [f, g]_a^b = 0 \text{ for each } f \in \mathcal{D}(A)\}.$$

The Sturm-Liouville differential expression (6.37) is called *regular at  $a$*  if  $a > -\infty$  and  $\{1/p_0, p_1, \dots, p_n, w\} \subset L^1(a, c)$  for any  $c \in (a, b)$ , and *singular at  $a$*  if it is not regular at  $a$ . The expression (6.37) is called *regular at  $b$*  if  $b < +\infty$  and  $\{1/p_0, p_1, \dots, p_n, w\} \subset L^1(c, b)$  for any  $c \in (a, b)$ , and *singular at  $b$*  if it is not regular at  $b$ . The expression (6.37) is called *regular* if it is regular at  $a$  and at

$b$ , and *singular* otherwise. If (6.37) is regular,  $2n$  boundary conditions are to be imposed to specify a selfadjoint extension of  $A_m$ . In the singular case the number of boundary conditions required to specify a selfadjoint extension of  $A_m$  could be any integer  $m \in \{0, 1, \dots, 2n\}$ .

**Example 6.10.** Let us consider the regular differential expression

$$(\tau f)(\mu) = (-1)^n f^{(2n)}(\mu), \quad \mu \in (-1, 1).$$

Then  $y^{[j]} = y^{(j)}$  and  $y^{[n+j]} = (-1)^j y^{(n+j)}$  ( $j = 0, 1, \dots, n$ ),  $p_0 \equiv 1$ ,  $p_1 = \dots = p_n \equiv 0$ , and  $w \equiv 1$ . Then the maximal domain  $\mathcal{D}(A_M)$  consists of all  $f \in L^2(-1, 1)$  for which all derivatives up to the  $(2n - 1)$ th order exist and are absolutely continuous on  $[-1, 1]$ . Further,

$$\begin{aligned} \langle \tau f, g \rangle - \langle f, \tau g \rangle &= (-1)^n \int_{-1}^1 \left\{ f^{(2n)}(\mu) \overline{g(\mu)} - f(\mu) \overline{g^{(2n)}(\mu)} \right\} d\mu \\ &= \left[ \sum_{j=1}^n (-1)^{n-j+1} \left( f^{(2n-j)} \overline{g^{(j-1)}} - f^{(j-1)} \overline{g^{(2n-j)}} \right) \right]_{-1}^1, \end{aligned}$$

as well as

$$\langle \tau f, f \rangle = \left[ \sum_{j=1}^n (-1)^{n-j+1} f^{(2n-j)} \overline{f^{(j-1)}} \right]_{-1}^1 + \int_{-1}^1 |f^{(n)}(\mu)|^2 d\mu.$$

Thus the boundary conditions  $y^{(j)}(-1) = y^{(j)}(1) = 0$  ( $j = 0, 1, \dots, n - 1$ ) define a selfadjoint operator  $A$  on  $L^2(-1, 1)$ , while  $\|A^{1/2}y\|_2 = \|y^{(n)}\|_2$ . Hence  $A$  is positive selfadjoint on  $L^2(-1, 1)$ . In [21, 22] it has been shown for the first time, though by different means, that the corresponding diffusion problem (6.4) is well posed.

## 6.4.2 Indefinite weight functions

Given a real weight function  $w \in L^1_{\text{loc}}(I)$  defined on an arbitrary bounded or unbounded interval  $I = (a, b)$  which has at most finitely many zeros, we consider the differential expression

$$(\tau f)(\mu) = \sum_{j=0}^n (-1)^j \left( p_{n-j}(\mu) f^{(j)}(\mu) \right)^{(j)}, \quad (6.39)$$

which is obtained from (6.37) by putting  $w(\mu) \equiv 1$  and, apart from having more leeway as to the sign of  $w(\mu)$ , satisfies conditions (a) and (b) [or, alternatively, (a') and (b')]. Let  $A$  be a nonnegative selfadjoint extension of the corresponding

minimal operator  $A_m$  on  $H = L^2(I; d\mu)$  and suppose  $A$  has a closed range. Let us define the injective selfadjoint operator  $T$  on  $H$  as follows:

$$\begin{cases} \mathcal{D}(T) = \{f \in L^2(I; d\mu) : wf \in L^2(I; d\mu)\}, \\ (Tf)(\mu) = w(\mu)f(\mu). \end{cases}$$

Then the completion  $H_T$  of  $\mathcal{D}(T)$  with respect to the scalar product  $\langle f, g \rangle_T = \langle Tf, g \rangle$  ( $f, g \in \mathcal{D}(T)$ ) is given by  $H_T = L^2(I; |w| d\mu)$ .

Now observe that the differential expression

$$(\tau f)(\mu) = \frac{1}{|w(\mu)|} \sum_{j=0}^n (-1)^j \left( p_{n-j}(\mu) f^{(j)}(\mu) \right)^{(j)}, \quad (6.40)$$

satisfies the conditions of Subsection 6.4.1. By suitably extending the corresponding minimal operator we obtain its selfadjoint extensions on  $H_T = L^2(I; |w| d\mu)$ , one of which coincides with  $|T|^{-1}A$ . We assume that  $T^{-1}A$  (or, equivalently,  $|T|^{-1}A$ ) has a closed range in  $H_T$ . Then the generalized eigenvector space  $Z_0$  of  $T^{-1}A$  at any zero eigenvalue satisfies

$$Z_0 = \text{Ker}(T^{-1}A)^2 \subset \mathcal{D}(T).$$

We make the technical assumption (obviously satisfied if  $w$  is bounded) that

$$H_A \stackrel{\text{def}}{=} \mathcal{D}(A^{1/2}) \subset \mathcal{D}(T). \quad (6.41)$$

Lemma 6.8 can now be proved as in the second-order case, albeit at the expense of additional technicalities. Although some details of the proof can already be found in [47], a full proof has only appeared in [48, Lemma 3.2]. In fact, for higher-order differential operators the functions  $m_j$  in the statement of Lemma 6.8 are to belong to  $C^n(\tau_j - \varepsilon, \tau_j + \varepsilon)$  and are to satisfy  $m_j(\tau_j) \neq 0$  and  $m'_j(\tau_j) = \dots = m_j^{(n-1)}(\tau_j) = 0$  for the operators  $X_\pm$  and  $Y_\pm$  to exist. Under these conditions the norm equivalence identity (6.36) is true. We may therefore conclude that Theorem 6.9 also holds in the higher-order case.

## Chapter 7

# Noncausal Continuous Time Systems

In this chapter we study various types of noncausal continuous time systems. Contrary to the usual continuous time systems obeying the equations<sup>1</sup>

$$\dot{x}(t) = -iAx(t) + Bu(t), \quad (7.1a)$$

$$y(t) = -iCx(t) + Du(t), \quad (7.1b)$$

where  $t \in \mathbb{R}^+$  is time,  $u(t)$  is input,  $y(t)$  is output,  $x(t)$  is the state, and  $-iA$  generates a strongly continuous semigroup, we now consider  $t \in \mathbb{R}$  and require  $-iA$  to be exponentially dichotomous. This amounts to dropping the causality assumption on the linear system. Various theories can be developed, parallelling existing theories for causal systems. In Section 7.1 we require  $-iA$  to be exponentially dichotomous and  $B$  and  $C$  to be bounded. This includes the direct generalization of finite-dimensional linear systems theory, where  $A$ ,  $B$ ,  $C$ , and  $D$  are all matrices and  $A$  does not have real eigenvalues. In Section 7.2 we pass to a formalism with two state spaces (one densely and continuously imbedded into the other), where the exponentially dichotomous operator  $-iA$  on the larger state space extends that on the smaller state space, the input operator  $B$  is bounded from the input space **into** the larger state space, and the output operator  $C$  is bounded **from** the smaller state space into the output space. Also adopting a complex Hilbert space setting, we thus obtain the so-called extended Pritchard-Salamon realizations. At the same time we discuss left and right Pritchard-Salamon realizations, where only one state space is used at the time. In either section we introduce the weighting pattern and the transfer function and derive their usual algebraic

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<sup>1</sup>In linear systems theory we usually deal with equations of the form  $\dot{x}(t) = Ax(t) + Bu(t)$  and  $y(t) = Cx(t) + Du(t)$ . The reason for this difference in notation is our preference to use Fourier transforms instead of Laplace transforms to arrive at the transfer function.

properties (inversion, multiplication, taking adjoints, and factorization). For left and right Pritchard-Salamon realizations all of these algebraic properties (except for the adjoint) are derived. We also discuss at length the realization problem of expressing the input-output map as a noncausal linear system of a certain type.

## 7.1 Noncausal state linear systems

In [52],  $\Sigma(A, B, C, D)$  is said to denote the state linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (7.2a)$$

$$y(t) = Cx(t) + Du(t), \quad (7.2b)$$

whenever  $A$  is the infinitesimal generator of a (not necessarily bounded) strongly continuous semigroup on a complex Hilbert space  $X$ ,  $B$  is a bounded linear operator from a complex Hilbert space  $U$  into  $X$ ,  $C$  is a bounded linear operator from  $X$  into a complex Hilbert space  $Y$ , and  $D \in \mathcal{L}(U, Y)$ . Given the input  $u \in L^2([0, \tau]; U)$ , the state is the mild solution of (7.2a), i.e.,

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds, \quad 0 \leq t \leq \tau,$$

and the output is given by (7.2b). In this section we define a noncausal state linear system by replacing  $A$  and  $C$  by  $-iA$  and  $-iC$  and letting  $-iA$  be exponentially dichotomous. The main difference will then be that time  $t$  can in principle run forward and backward (hence  $t \in \mathbb{R}$  instead of  $t \in \mathbb{R}^+$ ) and convolution integrals will involve convolution kernels supported in  $\mathbb{R}$  and not merely in  $\mathbb{R}^+$ .

### 7.1.1 Definitions, basic results, and adjoints

By a *noncausal state linear system* we mean the ordered  $(7 = 4 + 1 + 2)$ -tuple  $\theta = (A, B, C, D; X; U, Y)$ , where  $U$ ,  $X$ , and  $Y$  are complex Hilbert spaces called the *input space*, *state space*, and *output space*, respectively,  $-iA(X \rightarrow X)$  is exponentially dichotomous,  $B \in \mathcal{L}(U, X)$ ,  $C \in \mathcal{L}(X, Y)$ , and  $D \in \mathcal{L}(U, Y)$ . If  $u \in L^2(\mathbb{R}; U)$  is the *input*, then the *state*  $x(t)$  with *state impulse*  $x_0 \in X$  is the (unique) mild solution of (7.1a) in  $L^2(\mathbb{R}; X)$  under the jump condition  $x(0^+) - x(0^-) = x_0$ , i.e.,

$$x(t) = E(t; -iA)x_0 + \int_{-\infty}^{\infty} E(t-s; -iA)Bu(s)ds,$$

while the *output*  $y(t)$  is given by (7.1b).

Given the noncausal state linear system  $\theta = (A, B, C, D; X; U, Y)$ , by the *controllability map* we mean the linear operator  $\Gamma_\theta : L^2(\mathbb{R}; U) \rightarrow BC(\mathbb{R}; X)$  defined by

$$(\Gamma_\theta u)(t) = \int_{-\infty}^{\infty} E(t-s; -iA)Bu(s)ds, \quad u \in L^2(\mathbb{R}; U).$$

By the *observability map* we mean the linear operator  $\Lambda_\theta : X \rightarrow L^2(\mathbb{R}; Y)$  defined by

$$(\Lambda_\theta x)(t) = CE(t; -iA)x, \quad x \in X, \quad t \in \mathbb{R}.$$

Then  $\Gamma_\theta$  and  $\Lambda_\theta$  are well defined and bounded. We can then express the output  $y(t)$  in terms of the input  $u(t)$  and the state impulse  $x_0$  by

$$y(t) = Du(t) + \int_{-\infty}^{\infty} k_\theta(t-s)u(s)ds - iCE(t; -iA)x_0, \quad (7.3)$$

where

$$k_\theta(t) = -iCE(t; -iA)B$$

is called the *weighting pattern*. Equation (7.3) can be written in the short-hand form as

$$y = Du - iC\Gamma_\theta u - i\Lambda_\theta x_0,$$

where  $\Lambda_\theta$  and  $\Gamma_\theta$  are the observability and controllability maps. Then for  $\varepsilon \in [0, \omega_\theta)$  and  $x \in U$  we have  $e^{\varepsilon|\cdot|}k_\theta(\cdot)x \in L^2(\mathbb{R}; Y)$ , uniformly in  $x$  on bounded subsets of the input space  $U$ . Here  $-\omega_\theta$  denotes the exponential growth bound of the bisemigroup  $E(\cdot; -iA)$ .

Let us now define the *transfer function* of the noncausal state linear system  $\theta = (A, B, C, D; X; U, Y)$  as follows:

$$W_\theta(\lambda)x = Dx + \int_{-\infty}^{\infty} e^{i\lambda t}k_\theta(t)xdt = [D + C(\lambda - A)^{-1}B]x,$$

where  $x \in U$ . Then for the Fourier transforms of the input and output we have

$$\hat{y}(\lambda) = W_\theta(\lambda)\hat{u}(\lambda) + C(\lambda - A)^{-1}x_0,$$

where  $x_0$  is the state impulse. Thus for  $x_0 = 0$  the transfer function describes the input-output map upon Fourier transformation. Clearly, because there exists  $\varepsilon > 0$  such that  $\int_{-\infty}^{\infty} e^{\varepsilon|t|}\|k_\theta(t)x\|dt$  converges uniformly in  $x$  on bounded subsets of  $U$  while  $k_\theta$  is strongly continuous with a jump discontinuity in  $t = 0$ , the transfer function is analytic in  $C_\varepsilon = \{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \leq \varepsilon\}$  and tends to  $D$  in the strong sense as  $|\lambda| \rightarrow +\infty$  in  $C_\varepsilon$ .

Given the noncausal state linear system  $\theta = (A, B, C, D; X; U, Y)$ , it is clear that also

$$\theta^* \stackrel{\text{def}}{=} (A^*, C^*, B^*, D^*; X; Y, U)$$

is a noncausal state linear system. Moreover, its weighting pattern and transfer function satisfy

$$\begin{aligned} k_{\theta^*}(t) &= k_\theta(-t)^*, \\ W_{\theta^*}(\lambda) &= W_\theta(\bar{\lambda})^*. \end{aligned}$$

Let us now prove that  $(\theta^*)^* = \theta$ . This amounts to proving that

$$(\theta^*)^* = (A^{**}, B, C, D; X; U, Y) = (A, B, C, D; X; U, Y) = \theta,$$

or that  $A^{**} = A$ . Let  $E$  and  $E_{**}$  be the bisemigroups generated by  $-iA$  and  $-iA^{**}$ , respectively. Then for all  $\lambda$  in a strip about the real line and for all  $y, z \in Y$  we have

$$\int_{-\infty}^{\infty} e^{i\lambda t} \langle E(t)y, z \rangle dt = \int_{-\infty}^{\infty} e^{i\lambda t} \langle E_{**}(t)y, z \rangle dt,$$

which implies  $E = E_{**}$  and hence  $A = A^{**}$ . Thus  $(\theta^*)^* = \theta$  indeed.

### 7.1.2 Generating noncausal state linear systems

In this subsection we construct a noncausal state linear system  $\theta^\times$  from a given noncausal state linear system  $\theta$  such that

$$W_{\theta^\times}(\lambda) = W_\theta(\lambda)^{-1}, \quad \lambda \in \mathbb{R}.$$

We also construct the product  $\theta = \theta_1 \theta_2$  of two noncausal state linear systems  $\theta_1$  and  $\theta_2$  such that

$$W_\theta(\lambda) = W_{\theta_1}(\lambda)W_{\theta_2}(\lambda), \quad \lambda \in \mathbb{R}.$$

Finally, we study spectral factorizations of transfer functions of noncausal state linear systems.

The following three results are easily verified by inspection and their proofs are left to the reader. In a different context they are in fact well known. In linear systems theory they describe (i) the transfer function of the (noncausal) state linear system obtained by interchanging the roles of input and output, (ii) the transfer function of the system obtained by letting the output of the first system be the input of the second system, and (iii) the transfer functions of the systems obtained by decomposing a given system as a cascade of two systems, where one system is stable and the other is antistable. Similar results are valid when studying operator models on a complex Hilbert space [150, 36]. The general principles behind the three results have been explained from a common perspective in [15, 19, 18]. Before formulating the third result we modify the definitions of left and right canonical and quasi-canonical factorization given in Section 4.1 to deal with operator functions on the extended real line (rather than the extended imaginary line).

In the statements of Theorems 7.1, 7.3, and 7.4 the hypothesis that certain operators are exponentially dichotomous, can be replaced by the hypothesis that the resolvent sets of these operators contain a vertical strip about the imaginary axis and are uniformly bounded on this strip. This is immediate from Theorem 2.13. In Theorem 7.2 the exponential dichotomy of  $-iA$  follows from that of  $-iA_1$  and  $-iA_2$  and the boundedness of  $B_1C_2$ , also as a result of Theorem 2.13.



**Theorem 7.1.** *Let  $\theta = (A, B, C, D; X; U, Y)$  be a noncausal state linear system for which  $D$  is invertible. Suppose  $-iA^\times$ , where  $A^\times = A - BD^{-1}C$ , is exponentially dichotomous. Then*

$$\theta^\times = (A^\times, BD^{-1}, -D^{-1}C, D^{-1}; X; Y, U)$$

*is a noncausal state linear system and*

$$W_{\theta^\times}(\lambda) = W_\theta(\lambda)^{-1}, \quad \lambda \in \mathbb{R}.$$

**Theorem 7.2.** *Consider the following two noncausal state linear systems:*

$$\theta_1 = (A_1, B_1, C_1, D_1; X_1; Z, Y) \quad \text{and} \quad \theta_2 = (A_2, B_2, C_2, D_2; X_2; U, Z).$$

*Then  $\theta = (A, B, C, D; X; U, Y)$ , where  $X = X_1 \dot{+} X_2$ ,*

$$A = \begin{pmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 D_2 \\ B_2 \end{pmatrix}, \quad C = (C_1 \quad D_1 C_2), \quad D = D_1 D_2,$$

*is a noncausal state linear system. Moreover,*

$$W_\theta(\lambda) = W_{\theta_1}(\lambda)W_{\theta_2}(\lambda), \quad \lambda \in \mathbb{R}. \quad (7.4)$$

Let  $Y$  be a complex Banach space. Suppose  $W$  is an operator function defined on the extended real line with values in  $\mathcal{L}(Y)$ , which is continuous in the norm on  $\mathbb{R}$  and strongly continuous at  $\pm\infty$ . Then

$$W(\lambda) = W_+(\lambda)W_-(\lambda), \quad \lambda \in \mathbb{R} \cup \{\infty\},$$

is called a *left quasi-canonical factorization* of  $W$  with respect to the real line if:

1.  $W_\pm$  extends to an operator function that is continuous in the norm on  $\mathbb{C}^\pm \cup \mathbb{R}$ , analytic on  $\mathbb{C}^\pm$ , and strongly continuous on  $\overline{\mathbb{C}^\pm}$ .
2.  $W_\pm(\lambda)$  has a bounded inverse for all  $\lambda \in \overline{\mathbb{C}^\pm}$ .
3.  $W_\pm(\cdot)^{-1}$  is strongly continuous on  $\overline{\mathbb{C}^\pm}$ .

A factorization of  $W$  of the form

$$W(\lambda) = W_-(\lambda)W_+(\lambda), \quad \lambda \in \mathbb{R} \cup \{\infty\},$$

where the factors  $W_\pm$  have the properties 1–3 stated above, is called a *right quasi-canonical factorization* of  $W$  with respect to the real line. If  $W$  is assumed continuous in the norm on the extended real line and the continuity conditions in 1–3 hold with respect to the norm topology instead of the strong operator topology (thus making obsolete condition 3), the above factorizations are called *left* and *right canonical*.

**Theorem 7.3.** *Let  $\theta = (A, B, C, D; X; Y)$  be a noncausal state linear system, where  $D$  is invertible and  $-iA^\times = -i[A - BD^{-1}C]$ , is exponentially dichotomous. Suppose  $\Pi$  is a bounded projection on  $X$  such that*

$$\text{Ker } \Pi = \text{Im } E(0^+; -iA), \quad \text{Im } \Pi = \text{Im } E(0^-; -iA^\times).$$

*Partitioning the operators  $A$ ,  $B$ , and  $C$  with respect to the decomposition  $X = \text{Ker } \Pi \dot{+} \text{Im } \Pi$ , i.e.,*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = (C_1 \quad C_2),$$

*and writing  $D = D_1 D_2$  as the product of two invertible operators  $D_1$  and  $D_2$ , we get the left quasi-canonical factorization (7.4), where*

$$\begin{aligned} \theta_1 &= (A_{11}, B_1 D_2^{-1}, C_1, D_1; \text{Ker } \Pi; Y), \\ \theta_2 &= (A_{22}, B_2, D_1^{-1} C_2, D_2; \text{Im } \Pi; Y), \end{aligned}$$

*are noncausal state linear systems.*

If  $\Pi$  satisfies

$$\text{Ker } \Pi = \text{Im } E(0^+; -iA^\times), \quad \text{Im } \Pi = \text{Im } E(0^-; -iA),$$

then Theorem 7.3 leads to a right quasi-canonical factorization instead.

It is well known from linear systems theory and various operator models that left or right canonical factorizability of a transfer function of the form

$$W(\lambda) = D + C(\lambda - A)^{-1}B$$

with invertible  $D$  is equivalent to the validity of certain cross decompositions of the spectral subspaces of  $A$  and  $A^\times = A - BD^{-1}C$ . A full account of these results in various contexts can be found in [15, 18]. In Theorem 7.3 and the paragraph following its statement we have presented its counterpart for noncausal state linear systems. Below we connect this result to vector-valued convolution equations on the half-line (cf. [72], [68, Sec. I.8], [69, Ch. XIII]).

**Theorem 7.4.** *Let  $\theta = (A, B, C, D; X; Y)$  be a noncausal state linear system such that  $D$  is invertible and  $-iA^\times = -i[A - BD^{-1}C]$  is exponentially dichotomous. Then the following statements are equivalent:*

1.  $W_\theta(\cdot)$  has a left quasi-canonical factorization.
2. We have the decomposition

$$X = \text{Im } E(0^-; -iA) \dot{+} \text{Im } E(0^+; -iA^\times).$$

3. For every  $g \in L^2(\mathbb{R}^+; Y)$  the convolution equation

$$D\phi(t) + \int_0^\infty k_\theta(t-s)\phi(s)ds = g(t), \quad t \in \mathbb{R}^+, \quad (7.5)$$

*is uniquely solvable in  $L^2(\mathbb{R}^+; Y)$ .*

Moreover, we have the equivalent statements:

1'.  $W_\theta(\cdot)$  has a right quasi-canonical factorization.

2'. We have the decomposition

$$X = \operatorname{Im} E(0^+; -iA) \dot{+} \operatorname{Im} E(0^-; -iA^\times).$$

3'. For every  $g \in L^2(\mathbb{R}^-; Y)$  the convolution equation

$$D\phi(t) + \int_{-\infty}^0 k_\theta(t-s)\phi(s) ds = g(t), \quad t \in \mathbb{R}^-, \quad (7.6)$$

is uniquely solvable in  $L^2(\mathbb{R}^-; Y)$ .

We restrict ourselves to proving the equivalence of conditions 1–3. We omit the proof of the implication **(2)**  $\implies$  **(1)**, because the proof of the corresponding Theorem 7.3 can be given by inspection.

*Proof of Theorem 7.3.* **(1)**  $\implies$  **(3)** If  $W$  has a left quasi-canonical factorization

$$W_\theta(\lambda) = W_-(\lambda)W_+(\lambda),$$

then this factorization is necessarily valid for  $|\operatorname{Im} \lambda| \leq \varepsilon$ . By Fourier transformation we convert (7.5) into the Riemann-Hilbert problem

$$W_+(\lambda)\hat{\phi}_+(\lambda) + W_-(\lambda)^{-1}\hat{\phi}_-(\lambda) = W_-(\lambda)^{-1}\hat{g}(\lambda), \quad \lambda \in \mathbb{R},$$

where

$$\hat{\phi}_\pm(\lambda) = \pm \int_0^{\pm\infty} e^{i\lambda t} \phi(t) dt, \quad \hat{g}(\lambda) = \int_0^\infty e^{i\lambda t} g(t) dt,$$

and  $\phi(t) = -\int_0^\infty k_\theta(t-s)\phi(s) ds$  for  $t \in \mathbb{R}^-$ , the integral being understood as a Pettis integral. From the unique additive decomposition

$$W_-(\lambda)^{-1}\hat{g}(\lambda) = \hat{h}_+(\lambda) + \hat{h}_-(\lambda)$$

with  $\hat{h}_\pm(\lambda) = \pm \int_0^{\pm\infty} e^{i\lambda t} h(t) dt$  for some  $h \in L^2(\mathbb{R}; Y)$ , we find the unique solution

$$\phi(t) = \frac{1}{2\pi} \int_{-\infty}^\infty W_+(\lambda)^{-1} \hat{h}_+(\lambda) d\lambda, \quad t \in \mathbb{R}^+,$$

in  $L^2(\mathbb{R}; Y)$ .

**(3)**  $\implies$  **(2)** Let us define the linear operator

$$(\Xi_\theta \psi)(t) = -i \int_0^\infty E(t-s; -iA) B \psi(s) ds.$$

Then  $\Xi_\theta$  is a bounded linear operator from  $L^2(\mathbb{R}^+; Y)$  into  $L^2(\mathbb{R}^+; X)$ . Letting  $C$  stand for its natural extension to a bounded linear operator from  $L^2(\mathbb{R}^+; X)$  into  $L^2(\mathbb{R}^+; Y)$ , we have

$$T_\theta = D^{-1}C\Xi_\theta, \quad (7.7)$$

where

$$(T_\theta\phi)(t) = D^{-1} \int_0^\infty k_\theta(t-s)\phi(s) ds.$$

Thus for each  $h \in L^2(\mathbb{R}^+; X)$  the convolution equation

$$\psi(t) + \int_0^\infty E(t-s; -iA)BD^{-1}C\psi(s) ds = h(t), \quad t \in \mathbb{R}^+, \quad (7.8)$$

has a unique solution  $\psi \in L^2(\mathbb{R}^+; X)$ .

Given  $x \in X$ , we now consider the convolution equation (7.8) with right-hand side  $h(t) = E(t; -iA)x$  and write  $\psi(t) = G(t, x)$  for its solution. Then for  $u \geq 0$  and  $t > 0$  we have

$$\begin{aligned} G(t+u, x) - i \int_0^\infty E(t-s; -iA)BD^{-1}CG(s+u, x) ds \\ &= G(t+u, x) - i \int_u^\infty E(t+u-s; -iA)BD^{-1}CG(s, x) ds \\ &= E(t+u, -iA)x + i \int_0^u E(t+u-s; -iA)BD^{-1}CG(s, x) ds \\ &= E(t, -iA) \left[ E(u, -iA)x + i \int_0^u E(u-s; -iA)BD^{-1}CG(s, x) ds \right] \\ &= E(t, -iA) \left[ E(u, -iA)x + i \int_0^\infty E(u-s; -iA)BD^{-1}CG(s, x) ds \right] \\ &= E(t; -iA)G(u, x), \end{aligned}$$

where the penultimate transition follows from  $E(t; -iA)E(s; -iA) = 0$  for  $ts < 0$ . Thus we have derived the product rule

$$G(t+u, x) = G(t, G(u, x)), \quad x \in X, \quad t, u \in \mathbb{R}^+.$$

Now observe that for  $t \in \mathbb{R}^+$  (including  $t = 0^+$ ) the linear operator  $x \mapsto G(t, x)$  is bounded on  $X$ , which permits us to write  $G(t)x \stackrel{\text{def}}{=} G(t, x)$ . Because  $G(\cdot, x)$  is continuous as  $t \rightarrow 0^+$ , it follows that  $G(0)$  is a bounded projection on  $X$ . Further, for  $x \in X$  we have  $G(0, x) = 0$  iff  $G(t, x) \equiv 0$  iff  $E(t; -iA)x \equiv 0$  iff  $x \in \text{Im } E(0^-; -iA)$ . Thus,  $\text{Ker } G(0) = \text{Im } E(0^-; -iA)$ .

Substitute  $E(\cdot; -iA^\times)G(0)x$  with  $x \in X$  in the left-hand side of (7.8), yielding  $e(\cdot) \in L^2(\mathbb{R}^+; X)$ . Extending the convolution equation obtained to the full real line in the usual way and taking Fourier transforms we obtain

$$\hat{e}_+(\lambda) + \hat{e}_-(\lambda) = (\lambda - A)^{-1}(\lambda - A^\times).(\lambda - A^\times)^{-1}G(0)x = (\lambda - A)^{-1}G(0)x.$$

Hence,

$$\hat{e}_+(\lambda) = (\lambda - A)^{-1}(I_X + E(0^-; -iA))G(0)x = (\lambda - A)^{-1}(I_X + E(0^-; -iA))x,$$

the latter because of  $G(0)x \in \text{Im } E(0^-; -iA)$ . Consequently,

$$\text{Im } E(0^-; -iA) + \text{Im } G(0) = X, \quad G(0)[X] \subset \text{Im } E(0^+; -iA^\times).$$

Finally, from the unique solvability of (7.8) we conclude that

$$\text{Im } E(0^-; -iA) + \text{Im } E(0^+; -iA^\times) = X.$$

Moreover, we conclude that  $G(0)$  is the projection of  $X$  onto  $\text{Im } E(0^+; -iA^\times)$  along  $\text{Im } E(0^-; -iA)$ .  $\square$

Analogues of the following corollary have been proved for norm continuous operator functions belonging to certain (splitting and inverse closed) operator algebras [75], for the transfer function of the linear system naturally occurring in linear transport theory [77], and for transfer functions of systems constructed from block operators [134, 157]. For the latter results we refer the reader to Chapter 4.

**Corollary 7.5.** *Let  $\theta = (A, B, C, I_Y; X; Y)$  be a noncausal state linear system satisfying*

$$\sup_{|\text{Im } \lambda| \leq \mu} \|W_\theta(\lambda) - I_Y\|_{\mathcal{L}(Y)} < 1$$

*for some  $\mu \in (0, \omega_\theta)$ . Then  $W_\theta(\cdot)$  has a left and a right quasi-canonical factorization.*

The proof is based on the following norm estimate for the operator  $T_\theta$  defined by (7.7) (with  $D = I_Y$ ):

$$\|T_\theta\|_{\mathcal{L}(L^2(\mathbb{R}^+; Y))} \leq \sup_{\lambda \in \mathbb{R}} \|W_\theta(\lambda) - I_Y\|_{\mathcal{L}(Y)}.$$

Indeed, the convolution operator  $T_\theta$  can be represented as the product of the bounded linear operators depicted in the following commutative diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}; Y) & \xrightarrow{W_\theta(\cdot) - I_Y} & L^2(\mathbb{R}; Y) \\ \mathcal{F}_Y \uparrow & & \downarrow \mathcal{F}_Y^{-1} \\ L^2(\mathbb{R}; Y) & & L^2(\mathbb{R}; Y) \\ \text{imbedding} \uparrow & & \downarrow \text{projection} \\ L^2(\mathbb{R}^+; Y) & \xrightarrow{T_\theta} & L^2(\mathbb{R}^+; Y), \end{array}$$

where we make use of the unitarity of  $(2\pi)^{-1/2}\mathcal{F}_Y$  on  $L^2(\mathbb{R}; Y)$ ,  $\mathcal{F}_Y$  standing for the Fourier transform.

## 7.2 Extended Pritchard-Salamon realizations

In this section we discuss extended Pritchard-Salamon realizations. These were introduced in [97] as the forward-backward versions of the Pritchard-Salamon systems prevailing in the literature (cf. [129, 49, 50, 159, 96] and [122, Sec. 6.9]). In [96] the weighting patterns of Pritchard-Salamon systems have been studied in detail and the operator functions that can occur as transfer functions of a Pritchard-Salamon system have been characterized. However, as indicated by Mikkola [123], in [96] it has been incorrectly suggested that the weighting pattern is to be defined as a measurable operator-valued function. Instead, by defining it as a function from  $\mathbb{R} \times U$  to  $Y$ ,  $U$  and  $Y$  being the underlying input and output Hilbert spaces, the results of [96] still go through. In this section we develop the forward-backward version of [96] (essentially [97]) in a more transparent way. The increased transparency represents a different way of defining weighting patterns and a simplified treatment of adjoint extended Pritchard-Salamon realizations. Recently, Ball and Raney [12] introduced the discrete-time counterpart of extended Pritchard-Salamon realizations and applied their results to interpolation problems.

Extended Pritchard-Salamon realizations have two state spaces,  $V$  and  $W$ , one continuously and densely imbedded into the other by means of the imbedding  $\tau : W \rightarrow V$ . From such a realization two auxiliary realizations can be derived: a “left” Pritchard-Salamon realization with state space  $V$  and a “right” Pritchard-Salamon realization with state space  $W$ , both having the same transfer function and the same weighting pattern as the extended Pritchard-Salamon realization. We therefore first study these left and right Pritchard-Salamon realizations separately, disregarding the imbedding  $\tau$ . In fact, if  $Y$  has a finite dimension, the left realization was introduced before by Bart, Gohberg, and Kaashoek [16] and called BGK realization ever since. Here we also introduce its “dual” right realization. For either type, we generate other realizations of the same type as we did in Subsection 7.1.2. When discussing the adjoint realization and solving the realization problem we restrict ourselves to extended Pritchard-Salamon realizations.

### 7.2.1 Definitions and basic properties

In this subsection we give the definitions and basic properties of left, right, and extended PS-realizations and introduce their transfer functions, weighting patterns, and input-output operators.

**1. Left Pritchard-Salamon realizations.** We call

$$\theta_l = (A, B, \tilde{C}; V; Y)$$

a *left Pritchard-Salamon (left PS) realization* if the following conditions are fulfilled:

- L1.  $-iA(V \rightarrow V)$  is exponentially dichotomous.  
 L2.  $B \in \mathcal{L}(Y, V)$  and  $\tilde{C}(V \rightarrow Y)$  satisfies  $\mathcal{D}(\tilde{C}) \supset \mathcal{D}(A)$ . There are no boundedness or closedness assumptions on  $\tilde{C}$ , only its algebraic properties are being used.  
 L3. There exists a bounded linear operator  $\Lambda_{\theta_l} : V \rightarrow L^2(\mathbb{R}; Y)$  such that

$$\Lambda_{\theta_l} x = \tilde{C}E(\cdot; -iA)x, \quad x \in \mathcal{D}(A).$$

We may require that  $\mathcal{D}(\tilde{C}) = \mathcal{D}(A)$  without changing anything essential.

Letting  $-\omega_{\theta_l}$  denote the exponential growth bound of the bisemigroup  $E(\cdot; -iA)$ , we have

$$\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \omega_{\theta_l}\} \subset \rho(A)$$

and

$$(\lambda - A)^{-1}x = -i \int_{-\infty}^{\infty} e^{i\lambda t} E(t; -iA)x dt, \quad |\operatorname{Im} \lambda| < \omega_{\theta_l}, \quad x \in V. \quad (7.9)$$

**Lemma 7.6.** *Let  $\theta_l = (A, B, \tilde{C}; V; Y)$  be a left PS-realization. Then for every  $\mu \in [0, \omega_{\theta_l})$  there exists a constant  $\gamma(\mu)$  such that*

$$\left\| e^{\mu|\cdot|} \Lambda_{\theta_l} x \right\|_{L^2(\mathbb{R}; Y)} \leq \gamma(\mu) \|x\|_V, \quad x \in V. \quad (7.10)$$

*This statement remains valid if  $L^2$  is replaced by  $L^1$ .*

*Proof.* Let  $t_1 > 0$ ,  $\nu \in (-\omega_{\theta_l}, \omega_{\theta_l})$ , and  $\omega \in (|\nu|, \omega_{\theta_l})$ . Then for  $x \in \mathcal{D}(A)$  we have, for some finite constant  $c_1$ ,

$$\begin{aligned} & \int_0^{\infty} e^{2|\nu|s} \|(\Lambda_{\theta_l} x)(s)\|_Y^2 ds \\ & \leq \sum_{n=0}^{\infty} e^{2n|\nu|t_1} \int_0^{t_1} e^{2|\nu|s} \left\| \tilde{C}E(s; -iA)E(nt_1; -iA)x \right\|_Y^2 ds \\ & \leq e^{2|\nu|t_1} \|\Lambda_{\theta_l}\|^2 \sum_{n=0}^{\infty} e^{2n|\nu|t_1} \|E(nt_1; -iA)x\|_V^2 \leq c_1^2 \|x\|_V^2 \sum_{n=0}^{\infty} e^{-2n(\omega - |\nu|)t_1}. \end{aligned}$$

Similarly, we have, for  $x \in \mathcal{D}(A)$  and some finite constant  $c_2$ ,

$$\begin{aligned} & \int_{-\infty}^0 e^{2|\nu|s} \|(\Lambda_{\theta_l} x)(s)\|_Y^2 ds \\ & \leq \sum_{n=0}^{\infty} e^{2n|\nu|t_1} \int_0^{t_1} e^{2|\nu|s} \left\| \tilde{C}E(-s; -iA)E(-nt_1; -iA)x \right\|_Y^2 ds \\ & \leq e^{2|\nu|t_1} \|\Lambda_{\theta_l}\|^2 \sum_{n=0}^{\infty} e^{2n|\nu|t_1} \|E(-nt_1; -iA)x\|_V^2 \leq c_2^2 \|x\|_V^2 \sum_{n=0}^{\infty} e^{-2n(\omega - |\nu|)t_1}. \end{aligned}$$

Consequently,

$$\int_{-\infty}^{\infty} e^{2|\nu s|} \|(\Lambda_{\theta_l} x)(s)\|_Y^2 ds \leq (c_1^2 + c_2^2) \|x\|_V^2 \sum_{n=0}^{\infty} e^{-2n(\omega - |\nu|)t_1},$$

which implies (7.10).

The  $L^1$  part of this lemma is clear from the  $L^2$  part and the estimate

$$\int_{-\infty}^{\infty} e^{\mu|s|} \|\psi(s)\| ds \leq \left[ \int_{-\infty}^{\infty} e^{-2(\nu - \mu)|s|} ds \right]^{1/2} \left[ \int_{-\infty}^{\infty} e^{2\nu|s|} \|\psi(s)\|^2 ds \right]^{1/2} \quad (7.11)$$

whenever  $0 < \mu < \nu < \omega_{\theta_l}$ . □

As a result of the  $L^1$ -version of Lemma 7.6, we can now write

$$\tilde{C}(\lambda - A)^{-1}x = -i \int_{-\infty}^{\infty} e^{i\lambda t} (\Lambda_{\theta_l} x)(t) dt,$$

where  $x \in V$  and  $|\operatorname{Im} \lambda| < \omega_{\theta_l}$ . Thus,  $\tilde{C}(\lambda - A)^{-1}$  is bounded as a linear operator from  $V$  into  $Y$ , uniformly in  $\lambda$  on each horizontal strip  $|\operatorname{Im} \lambda| \leq \varepsilon$ , where  $\varepsilon \in (0, \omega_{\theta_l})$ . Also, it vanishes in the strong operator topology as  $|\lambda| \rightarrow +\infty$  within such a strip.

We now define the *transfer function*  $W_{\theta_l}$  of a left PS-realization by

$$W_{\theta_l}(\lambda) = I_Y + \tilde{C}(\lambda - A)^{-1}B,$$

where  $|\operatorname{Im} \lambda| < \omega_{\theta_l}$ . We then write the transfer function in the form

$$[W_{\theta_l}(\lambda) - I_Y]y = -i \int_{-\infty}^{\infty} e^{i\lambda t} (\Lambda_{\theta_l} B y)(t) dt,$$

where  $y \in Y$  and  $|\operatorname{Im} \lambda| < \omega_{\theta_l}$ . We also define the *weighting pattern* as the function  $k_{\theta_l} : \mathbb{R} \times Y \rightarrow Y$  given by

$$k_{\theta_l}(t, y) = -i(\Lambda_{\theta_l} B y)(t).$$

Then  $k_{\theta_l}(\cdot, y) \in L^2(\mathbb{R}; Y)$  for every  $y \in Y$ . For each  $y \in Y$  we then have

$$[W_{\theta_l}(\lambda) - I_Y]y = \int_{-\infty}^{\infty} e^{i\lambda t} k_{\theta_l}(t, y) dt, \quad |\operatorname{Im} \lambda| < \omega_{\theta_l}.$$

Let us now define the *input-output operator* as follows:

$$(T_{\theta_l} \phi)(t) = \int_{-\infty}^{\infty} k_{\theta_l}(t - s, \phi(s)) ds, \quad (7.12)$$



where the integral in (7.12) is a Pettis integral. Since

$$\left\| e^{\mu|\cdot|} k_{\theta_l}(\cdot, y) \right\|_{L^1(\mathbb{R}; Y)} \leq \gamma(\mu) \|B\|_{\mathcal{L}(Y, V)} \|y\|_Y$$

for every  $\mu \in [0, \omega_{\theta_l})$ , we have, by virtue of Lemma 2.11,

$$\left\| e^{\mu|\cdot|} T_{\theta_l} \phi \right\|_{L^2(\mathbb{R}; Y)} \leq \gamma(\mu) \|B\|_{\mathcal{L}(Y, V)} \|e^{\mu|\cdot|} \phi\|_{L^2(\mathbb{R}; Y)}.$$

Using the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}; Y) & \xrightarrow{T_{\theta_l}} & L^2(\mathbb{R}; Y) \\ \mathcal{F}_Y \downarrow & & \downarrow \mathcal{F}_V \\ L^2(\mathbb{R}; Y) & \xrightarrow{W_{\theta_l}(\cdot) - I_Y} & L^2(\mathbb{R}; Y), \end{array}$$

where  $\mathcal{F}_Y$  is the Fourier transform map on  $L^2(\mathbb{R}; Y)$ , we easily prove that

$$\|T_{\theta_l}\|_{L^2(\mathbb{R}; Y)} = \sup_{\lambda \in \mathbb{R}} \|W_{\theta_l}(\lambda) - I_Y\|.$$

**2. Right Pritchard-Salamon realizations.** We call

$$\theta_r = (A_W, \tilde{B}, C; W; Y)$$

a *right Pritchard-Salamon (right PS) realization* if the following conditions are fulfilled:

- R1.  $-iA_W(W \rightarrow W)$  is exponentially dichotomous.
- R2.  $C \in \mathcal{L}(W, Y)$ , and  $\tilde{B} : Y \rightarrow W$  is linear and defined on all of  $Y$ . There are no boundedness or closedness assumptions on  $\tilde{B}$ , only its algebraic properties are being used.
- R3. There exists a bounded linear operator  $\Gamma_{\theta_r} : L^2(\mathbb{R}; Y) \rightarrow W$  such that

$$\Gamma_{\theta_r} \phi = \int_{-\infty}^{\infty} E(s; -iA_W) \tilde{B} \phi(s) ds,$$

where  $\phi(t) = \sum_{j=1}^n \varphi_j(t) y_j$  for all  $n \in \mathbb{N}$ ,  $\varphi_1, \dots, \varphi_n \in L^2(\mathbb{R}; Y)$ , and  $y_1, \dots, y_n \in Y$ .

The integral in Condition R3 can be written as

$$\Gamma_{\theta_r} \phi = \sum_{j=1}^n \int_{-\infty}^{\infty} \varphi_j(s) E(s; -iA_W) \tilde{B} y_j ds,$$

which is a Bochner integral. We do not attach any meaning to the integral in Condition R3 for arbitrary  $\phi \in L^2(\mathbb{R}; Y)$ .

Letting  $-\omega_{\theta_r}$  denote the exponential growth bound of the bisemigroup  $E(\cdot; -iA_W)$ , we have

$$\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \omega_{\theta_r}\} \subset \rho(A_W)$$

and

$$(\lambda - A_W)^{-1}x = -i \int_{-\infty}^{\infty} e^{i\lambda t} E(t; -iA_W)x dt, \quad |\operatorname{Im} \lambda| < \omega_{\theta_r}, \quad x \in W. \quad (7.13)$$

**Lemma 7.7.** *Let  $\theta_r = (A_W, \tilde{B}, C; W; Y)$  be a right PS-realization. Then for every  $\mu \in [0, \omega_{\theta_r})$  there exists a constant  $\beta(\mu)$  such that*

$$\|\Gamma_{\theta_r} \phi\|_W \leq \beta(\mu) \|e^{\mu|\cdot|} \phi\|_{L^2(\mathbb{R}; Y)}, \quad e^{\mu|\cdot|} \phi \in L^2(\mathbb{R}; Y).$$

*This statement remains valid if  $L^2$  is replaced by  $L^1$ .*

*Proof.* Let  $t_1 > 0$ ,  $\nu \in (-\omega_{\theta_r}, \omega_{\theta_r})$ , and  $\omega \in (|\nu|, \omega_{\theta_r})$ . Let  $\phi(t) = \sum_{j=1}^n \varphi_j(t) y_j$ , where  $\varphi_1, \dots, \varphi_n \in L^2(\mathbb{R}; Y)$  and  $y_1, \dots, y_n \in Y$ , and put  $\phi^\#(s) = \phi(-s)$ . Writing

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{\nu s} E(s; -iA_W) \tilde{B} \phi(s) ds \\ &= \sum_{n=0}^{\infty} \left[ e^{n\nu t_1} E(nt_1; -iA_W) \int_0^{t_1} e^{\nu s} E(s; -iA_W) \tilde{B} \phi(nt_1 + s) ds \right. \\ & \quad \left. + e^{-n\nu t_1} E(-nt_1; -iA_W) \int_{-t_1}^0 e^{\nu s} E(s; -iA_W) \tilde{B} \phi(s - nt_1) ds \right], \end{aligned}$$

we estimate

$$\begin{aligned} & \left\| \int_{-\infty}^{\infty} e^{\nu s} E(s; -iA_W) \tilde{B} \phi(s) ds \right\|_W \\ & \leq \text{const.} \sum_{n=0}^{\infty} e^{-nt_1(\omega_{\theta_r} - |\nu|)} \left\| \int_0^{t_1} e^{\nu s} E(s; -iA_W) \tilde{B} \phi(s + nt_1) ds \right. \\ & \quad \left. + \int_{-t_1}^0 e^{\nu s} E(s; -iA_W) \tilde{B} \phi(s - nt_1) ds \right\|_W \leq \text{const.} \|\phi\|_{L^2(\mathbb{R}; Y)}, \end{aligned}$$

as claimed.

The  $L^1$  part of this lemma is clear from the  $L^2$  part and the estimate (7.11) whenever  $0 < \mu < \nu < \omega_{\theta_r}$ .  $\square$

Consider  $\phi = \varphi(\cdot)y \in L^2(\mathbb{R}; Y)$ , where  $y \in Y$  and  $\varphi \in L^2(\mathbb{R})$ . Then for each  $w \in W$  we have

$$\int_{-\infty}^{\infty} \langle E(s; -iA_W) \tilde{B} \phi(s), w \rangle ds = \int_{-\infty}^{\infty} \varphi(s) \langle E(s; -iA_W) \tilde{B} y, w \rangle ds,$$

so that

$$\left| \int_{-\infty}^{\infty} \langle E(s; -iA_W) \tilde{B} \phi(s), w \rangle ds \right| \leq \|\Gamma_{\theta_r}\| \|y\|_Y \|\varphi\|_2 \|w\|_W.$$

By the Riesz Representation Theorem (applied on  $L^2(\mathbb{R})$ ), this implies that

$$\left[ \int_{-\infty}^{\infty} \left| \langle E(s; -iA_W) \tilde{B} y, w \rangle \right|^2 ds \right]^{1/2} \leq \|\Gamma_{\theta_r}\| \|y\|_Y \|w\|_W.$$

In the same way we prove that, for  $\mu \in [0, \omega_{\theta_r})$ ,

$$\left[ \int_{-\infty}^{\infty} e^{2\mu|s|} \left| \langle E(s; -iA_W) \tilde{B} y, w \rangle \right|^2 ds \right]^{1/2} \leq \beta(\mu) \|y\|_Y \|w\|_W.$$

Using that

$$\langle (\lambda - A_W)^{-1} \tilde{B} y, w \rangle = -i \int_{-\infty}^{\infty} e^{i\lambda t} \langle E(s; -iA_W) \tilde{B} y, w \rangle ds,$$

we obtain for  $\lambda \in \mathbb{R} \pm i\mu$ , with  $\mu \in [0, \omega_{\theta_r})$ ,

$$\begin{aligned} \left| \langle (\lambda - A_W)^{-1} \tilde{B} y, w \rangle \right| &= \int_{-\infty}^{\infty} e^{\mu|s|} \left| \langle E(s; -iA_W) \tilde{B} y, w \rangle \right| ds \\ &\leq \text{const.} \|y\|_Y \|w\|_W, \end{aligned} \tag{7.14}$$

where the constant depends on  $\mu$ . Thus  $(\lambda - A_W)^{-1} \tilde{B}$  is a bounded linear operator from  $Y$  into  $W$ , uniformly in  $|\text{Im } \lambda| \leq \mu$ , irrespective of the choice of  $\mu \in (0, \omega_{\theta_r})$ . Also, it vanishes in the strong operator topology as  $|\lambda| \rightarrow +\infty$  within such a strip. Therefore, in condition R2 we can actually require that  $\tilde{B} : Y \rightarrow \text{Im } A_W$  without changing anything essential.

We can actually prove the existence, for each  $w \in W$ , of a vector function  $F(\cdot, w)$  such that

$$[(\bar{\lambda} - A_W)^{-1} \tilde{B}]^* w = -i \int_{-\infty}^{\infty} e^{i\lambda t} F(t, w) dt, \quad |\text{Im } \lambda| < \omega_{\theta_r}, \tag{7.15}$$

while for each  $\mu \in (0, \omega_{\theta_r})$ ,

$$\left\| e^{\mu|\cdot|} F(\cdot, w) \right\|_{L^2(\mathbb{R}; Y)} \leq \beta(\mu) \|w\|_W, \quad w \in W. \tag{7.16}$$

Indeed,

$$|\langle \Gamma_{\theta_r} \phi, w \rangle| \leq \|\Gamma_{\theta_r}\| \|\phi\|_{L^2(\mathbb{R}; Y)} \|w\|_W,$$

so that  $\phi \mapsto \langle \Gamma_{\theta_r} \phi, w \rangle$  is a bounded linear functional on  $L^2(\mathbb{R}; Y)$  with norm bounded above by  $\|\Gamma_{\theta_r}\| \|w\|_W$ . Thus, by the Riesz Representation Theorem, for

each  $w \in W$  there exists  $F(\cdot, w) \in L^2(\mathbb{R}; Y)$  of norm bounded above by  $\|\Gamma_{\theta_r}\| \|w\|_W$  such that

$$\langle \Gamma_{\theta_r} \phi, w \rangle_W = \langle \phi, F(\cdot, w) \rangle_{L^2(\mathbb{R}; Y)}, \quad \phi \in L^2(\mathbb{R}; Y), \quad w \in W.$$

Using Lemma 7.7 we easily prove (7.15) and (7.16) by applying the Riesz Representation Theorem in  $L^2(\mathbb{R}, e^{2\mu|t|} dt; Y)$ .

We now define the *transfer function*  $W_{\theta_r}$  of a right PS-realization by

$$W_{\theta_r}(\lambda) = I_Y + C(\lambda - A_W)^{-1} \tilde{B},$$

where  $|\operatorname{Im} \lambda| < \omega_{\theta_r}$ . We then write the transfer function in the form

$$[W_{\theta_r}(\lambda) - I_Y]y = -i \int_{-\infty}^{\infty} e^{i\lambda t} CE(t; -iA_W) \tilde{B}y \, dt,$$

where  $y \in Y$  and  $|\operatorname{Im} \lambda| < \omega_{\theta_r}$ . We also define the *weighting pattern* as the function  $k_{\theta_r} : \mathbb{R} \times Y \rightarrow Y$  given by

$$k_{\theta_r}(t, y) = -iCE(t; -iA_W) \tilde{B}y. \quad (7.17)$$

Then  $k_{\theta_r}(\cdot, y) \in L^2(\mathbb{R}; Y)$  for each  $y \in Y$ . For each  $y \in Y$  we then have

$$[W_{\theta_r}(\lambda) - I_Y]y = \int_{-\infty}^{\infty} e^{i\lambda t} k_{\theta_r}(t, y) \, dt, \quad |\operatorname{Im} \lambda| < \omega_{\theta_r}. \quad (7.18)$$

Let us now define the *input-output operator* as follows:

$$(T_{\theta_r} \phi)(t) = \int_{-\infty}^{\infty} k_{\theta_r}(t - s, \phi(s)) \, ds,$$

where  $\phi(t) = \sum_{j=1}^n \varphi_j(t) y_j$  for certain scalar functions  $\varphi_1, \dots, \varphi_n \in L^2(\mathbb{R})$  and vectors  $y_1, \dots, y_n \in Y$ . Using the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}; Y) & \xrightarrow{T_{\theta_r}} & L^2(\mathbb{R}; Y) \\ \mathcal{F}_Y \downarrow & & \downarrow \mathcal{F}_Y \\ L^2(\mathbb{R}; Y) & \xrightarrow{W_{\theta_r}(\cdot) - I_Y} & L^2(\mathbb{R}; Y), \end{array}$$

where  $\mathcal{F}_Y$  is the Fourier transform map on  $L^2(\mathbb{R}; Y)$ , we can extend  $T_{\theta_r}$  to a bounded linear operator on  $L^2(\mathbb{R}; Y)$  satisfying

$$\|T_{\theta_r}\|_{L^2(\mathbb{R}; Y)} = \sup_{\lambda \in \mathbb{R}} \|W_{\theta_r}(\lambda) - I_Y\|.$$

With the help of a similar diagram we can employ (7.14) and obtain

$$\left\| e^{\mu|\cdot|} T_{\theta_r} \phi \right\|_{L^2(\mathbb{R}; Y)} \leq \beta(\mu) \|C\|_{\mathcal{L}(W, Y)} \|e^{\mu|\cdot|} \phi\|_{L^2(\mathbb{R}; Y)}.$$

**3. Extended Pritchard-Salamon realizations.** Let  $V$  be a complex Hilbert space and  $A(V \rightarrow V)$  a linear operator with domain  $\mathcal{D}(A)$ . Suppose  $W$  is another complex Hilbert space and  $\tau : W \rightarrow V$  is a continuous and dense imbedding. Then the linear operator  $A_W(W \rightarrow W)$  defined by

$$\begin{cases} \mathcal{D}(A_W) = \{x \in W : \tau x \in \mathcal{D}(A), A\tau x \in \tau[W]\}, \\ \tau A_W x = A\tau x, \end{cases}$$

is called the *part* of  $W$  in  $V$  (with respect to  $\tau$ ). Then it is easily shown that  $A_W$  is a closed operator whenever  $A$  is a closed operator. Consequently, if  $\mathcal{D}(A) = V$ ,  $A \in \mathcal{L}(V)$ , and  $A\tau[W] \subset \tau[W]$ , then  $\mathcal{D}(A_W) = W$  and hence, by the Closed Graph Theorem,  $A_W \in \mathcal{L}(W)$ .

If  $\rho(A) \cap \rho(A_W) \neq \emptyset$ , it is easily shown that  $A_W$  is the part of  $A$  in  $W$  if and only if for some (and hence all)  $\lambda \in \rho(A) \cap \rho(A_W)$  we have the intertwining relation

$$(\lambda - A)^{-1}\tau = \tau(\lambda - A_W)^{-1}. \quad (7.19)$$

In the rest of this chapter we shall use this characterization of  $A_W$  instead of its actual definition.

We call

$$\theta = (A, B, C; V, W, \tau; Y) \quad (7.20)$$

an *extended Pritchard-Salamon (PS) realization* if the following conditions are fulfilled:

- E1.  $-iA(V \rightarrow V)$  is exponentially dichotomous.
- E2. The part  $-iA_W$  of  $-iA$  in  $W$  is exponentially dichotomous.
- E3.  $B \in \mathcal{L}(Y, V)$  and  $C \in \mathcal{L}(W, Y)$ .
- E4. There exists a bounded linear operator  $\Lambda_\theta : V \rightarrow L^2(\mathbb{R}; Y)$  such that

$$\Lambda_\theta \tau x = CE(\cdot; -iA_W)x, \quad x \in W.$$

- E5. There exists a bounded linear operator  $\Gamma_\theta : L^2(\mathbb{R}; Y) \rightarrow W$  such that

$$\tau \Gamma_\theta \phi = \int_{-\infty}^{\infty} E(s; -iA) B \phi(s) ds \quad (\in \tau[W]), \quad \phi \in L^2(\mathbb{R}; Y).$$

Throughout this section  $-\omega_\theta$  will denote the maximum of the exponential growth bounds of the bisemigroups  $E(\cdot; -iA)$  and  $E(\cdot; -iA_W)$ . Then conditions E1–E2 imply that

$$\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \omega_\theta\} \subset \rho(A) \cap \rho(A_W).$$

For  $\lambda$  in this strip we immediately have (7.9) and (7.13). Using (7.19) it is easily shown that

$$E(t; -iA)\tau = \tau E(t; -iA_W), \quad 0 \neq t \in \mathbb{R} \text{ or } t = 0^\pm. \quad (7.21)$$

We shall prove shortly that any extended PS-realization (7.20) leads to a left PS-realization and a right PS-realization having the same transfer function, the same weighting pattern, and the same input-output operator. To derive the conditions L3 and R3 in the definitions of a left PS-realization and a right PS-realization, we now prove the following analogue of Lemmas 7.6 and 7.7.

**Lemma 7.8.** *Let  $\theta = (A, B, C; V, W, \tau; Y)$  be an extended PS-realization. Then for every  $\mu \in [0, \omega_\theta)$  there exist constants  $\beta(\mu)$  and  $\gamma(\mu)$  such that*

$$\left\| e^{\mu|\cdot|} \Lambda_\theta x \right\|_{L^2(\mathbb{R}; Y)} \leq \gamma(\mu) \|x\|_V, \quad x \in V; \quad (7.22)$$

$$\|\Gamma_\theta \phi\|_W \leq \beta(\mu) \|e^{\mu|\cdot|} \phi\|_{L^2(\mathbb{R}; Y)}, \quad e^{\mu|\cdot|} \phi \in L^2(\mathbb{R}; Y). \quad (7.23)$$

*These statements remain valid if  $L^2$  is replaced by  $L^1$ .*

The proof of Lemma 7.8 is similar to the combined proofs of Lemmas 7.6 and 7.7. However, subtle changes with respect to these two preceding proofs have convinced us to give the full proof of Lemma 7.8.

*Proof.* Let  $t_1 > 0$ ,  $\nu \in (-\omega_\theta, \omega_\theta)$ , and  $\omega \in (|\nu|, \omega_\theta)$ . Then for  $x = \tau y$  with  $y \in W$ , we have for some finite constant  $c_1$ ,

$$\begin{aligned} & \int_0^\infty e^{2|\nu|s} \|(\Lambda_\theta \tau y)(s)\|_Y^2 ds \\ & \leq \sum_{n=0}^\infty e^{2n|\nu|t_1} \int_0^{t_1} e^{2|\nu|s} \|CE(s; -iA_W)E(nt_1; -iA_W)y\|_Y^2 ds \\ & \leq e^{2|\nu|t_1} \|\Lambda_\theta\|^2 \sum_{n=0}^\infty e^{2n|\nu|t_1} \|E(nt_1; -iA)\tau y\|_V^2 \leq c_1^2 \|\tau y\|_V^2 \sum_{n=0}^\infty e^{-2n(\omega-|\nu|)t_1}. \end{aligned}$$

Similarly, we have, for some finite constant  $c_2$ ,

$$\begin{aligned} & \int_{-\infty}^0 e^{2|\nu|s} \|(\Lambda_\theta \tau y)(s)\|_Y^2 ds \\ & \leq \sum_{n=0}^\infty e^{2n|\nu|t_1} \int_0^{t_1} e^{2|\nu|s} \|CE(-s; -iA_W)E(-nt_1; -iA_W)y\|_Y^2 ds \\ & \leq e^{2|\nu|t_1} \|\Lambda_\theta\|^2 \sum_{n=0}^\infty e^{2n|\nu|t_1} \|E(-nt_1; -iA)\tau y\|_V^2 \leq c_2^2 \|\tau y\|_V^2 \sum_{n=0}^\infty e^{-2n(\omega-|\nu|)t_1}. \end{aligned}$$

Consequently,

$$\int_{-\infty}^\infty e^{2|\nu|s} \|(\Lambda_\theta \tau y)(s)\|_Y^2 ds \leq (c_1^2 + c_2^2) \|\tau y\|_V^2 \sum_{n=0}^\infty e^{-2n(\omega-|\nu|)t_1},$$

which implies (7.22).

By the same token, let  $\phi \in L^2(\mathbb{R}; Y)$  and put  $\phi^\#(s) = \phi(-s)$ . Writing

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{\nu s} E(s; -iA) B \phi(s) ds \\ &= \sum_{n=0}^{\infty} \left[ e^{n\nu t_1} E(nt_1; -iA) \int_0^{t_1} e^{\nu s} E(s; -iA) B \phi(nt_1 + s) ds \right. \\ & \quad \left. + e^{-n\nu t_1} E(-nt_1; -iA) \int_{-t_1}^0 e^{\nu s} E(s; -iA) B \phi(s - nt_1) ds \right], \end{aligned}$$

we estimate

$$\begin{aligned} & \left\| \tau^{-1} \underbrace{\int_{-\infty}^{\infty} e^{\nu s} E(s; -iA) B \phi(s) ds}_{\in \tau[W]} \right\|_W \\ & \leq \text{const.} \sum_{n=0}^{\infty} e^{-nt_1(\omega_\theta - |\nu|)} \left\| \tau^{-1} \underbrace{\int_0^{t_1} e^{\nu s} E(s; -iA) B \phi(s + nt_1) ds}_{\in \tau[W]} \right. \\ & \quad \left. + \underbrace{\tau^{-1} \int_{-t_1}^0 e^{\nu s} E(s; -iA) B \phi(s - nt_1) ds}_{\in \tau[W]} \right\|_W \leq \text{const.} \|\phi\|_{L^2(\mathbb{R}; Y)}, \end{aligned}$$

which proves (7.23).

The  $L^1$  part of this lemma is clear from the  $L^2$  part and the estimate (7.11) whenever  $0 < \mu < \nu < \omega_\theta$ .  $\square$

We now relate extended PS-realizations to left and right PS-realizations by deriving the following result.

**Theorem 7.9.** *Let  $\theta = (A, B, C; V, W, \tau; Y)$  be an extended PS-realization. Then there exist a unique linear operator  $\tilde{C} : \mathcal{D}(A) \rightarrow Y$  and a unique linear operator  $\tilde{B} : Y \rightarrow \text{Im } A_W$  such that*

$$C(\lambda - A_W)^{-1} = \tilde{C}(\lambda - A)^{-1}\tau, \quad |\text{Im } \lambda| < \omega_\theta, \quad (7.24)$$

$$\tau(\lambda - A_W)^{-1}\tilde{B} = (\lambda - A)^{-1}B, \quad |\text{Im } \lambda| < \omega_\theta. \quad (7.25)$$

Moreover,  $\theta_l = (A, B, \tilde{C}; V; Y)$  and  $\theta_r = (A_W, \tilde{B}, C; W; Y)$  are left and right PS-realizations. Conversely, let  $\theta_l = (A, B, \tilde{C}; V; Y)$  be a left PS-realization,  $\theta_r = (A_W, \tilde{B}, C; W; Y)$  a right PS-realization, and  $\tau : W \rightarrow V$  a continuous and dense imbedding such that (7.19), (7.24), and (7.25) are satisfied. Then  $\theta = (A, B, C; V, W, \tau; Y)$  is an extended PS-realization.

Observe that Theorem 7.9 does not state that  $\tilde{C}$  and  $\tilde{B}$  are bounded or even closed operators, just that they exist in the algebraic sense. In [97] only the construction of  $\tilde{C}$  is given, not that of  $\tilde{B}$ .

*Proof.* According to Lemma 7.8 we have, for each  $\mu \in [0, \omega_\theta)$ ,

$$\int_{-\infty}^{\infty} e^{\mu t} \|CE(t; -iA_W)x\| dt \leq \tilde{\gamma}(\mu) \|\tau x\|_V.$$

Taking the Fourier transform we get, for each  $\mu \in [0, \omega_\theta)$ ,

$$\|C(\lambda - A_W)^{-1}x\| \leq \tilde{\gamma}(\mu) \|\tau x\|_V, \quad |\operatorname{Im} \lambda| \leq \mu.$$

Thus, for these  $\lambda$ ,  $C(\lambda - A_W)^{-1}\tau^{-1}$  extends to a bounded linear operator,  $\tilde{C}(\lambda)$ , from  $V$  into  $Y$ . Hence there exists a unique linear operator  $\tilde{C} : \mathcal{D}(A) \rightarrow Y$  such that

$$\tilde{C}(\lambda) = \tilde{C}(\lambda - A)^{-1}, \quad |\operatorname{Im} \lambda| < \omega_\theta.$$

We now easily check that for  $x \in \mathcal{D}(A_W)$  (and hence  $\tau x \in \mathcal{D}(A)$ )

$$(\Lambda_{\theta_l} \tau x)(t) = CE(t; -iA_W)x = (\Lambda_\theta \tau x)(t),$$

so that  $\theta_l$  satisfies Conditions L1–L3.

Next, for every  $t \in \mathbb{R}$ ,  $\mu \in [0, \omega_\theta)$ , and  $\phi \in L^2(\mathbb{R}; Y)$  we have

$$\left\| \tau^{-1} \underbrace{\int_{-\infty}^{\infty} e^{\mu|s|} E(s; -iA) B \phi(t-s) ds}_{\in \tau[W]} \right\|_W \leq c(\mu) \|\phi\|_{L^2(\mathbb{R}; Y)}.$$

Applying the Fourier transform we see that

$$\left\| \tau^{-1} \underbrace{(\lambda - A)^{-1} B y}_{\in \tau[W]} \right\|_W \leq c(\mu) \|y\|_Y, \quad |\operatorname{Im} \lambda| \leq \mu.$$

Thus, for these  $\lambda$ ,  $\tau^{-1}(\lambda - A)^{-1}B$  extends to a bounded linear operator,  $\tilde{B}(\lambda)$ , from  $Y$  into  $W$ . Hence there exists a unique linear operator  $\tilde{B} : Y \rightarrow \operatorname{Im} A_W$  such that

$$\tilde{B}(\lambda) = (\lambda - A_W)^{-1} \tilde{B},$$

as claimed.



We now easily check that for  $\phi \in L^2(\mathbb{R}; Y)$  satisfying  $\phi(t) = \sum_{j=1}^n \varphi_j(t)y_j$  for certain vectors  $y_1, \dots, y_n \in Y$  and  $\varphi_1, \dots, \varphi_n \in L^2(\mathbb{R})$ ,

$$\tau \Gamma_{\theta_r} \phi = \int_{-\infty}^{\infty} \tau E(s; -iA_W) \tilde{B} \phi(s) ds = \int_{-\infty}^{\infty} E(s; -iA) B \phi(s) ds = \tau \Gamma_{\theta} \phi,$$

so that  $\theta_r$  satisfies Conditions R1–R3.

Conversely, if  $\theta_l$  satisfies Conditions L1–L3,  $\theta_r$  satisfies Conditions R1–R3, and (7.19), (7.24), and (7.25) are satisfied, then we easily verify that

$$\Lambda_{\theta} \tau = \Lambda_{\theta_l} \tau, \quad \tau \Gamma_{\theta} = \tau \Gamma_{\theta_r},$$

on suitable dense linear subspaces. As a result,  $A_W$  is the part of  $A$  in  $W$  (cf. (7.19)) and  $\theta$  is an extended PS-realization.  $\square$

For  $\lambda$  satisfying  $|\operatorname{Im} \lambda| < \omega_{\theta}$  we have

$$\tilde{C}(\lambda - A)^{-1} B \stackrel{(7.25)}{=} \tilde{C} \tau (\lambda - A_W)^{-1} \tilde{B} \stackrel{(7.19)}{=} \tilde{C}(\lambda - A)^{-1} \tau \tilde{B} \stackrel{(7.24)}{=} C(\lambda - A_W)^{-1} \tilde{B}.$$

We can therefore define the *transfer function*  $W_{\theta}$  of an extended PS-realization by either of the equivalent expressions

$$W_{\theta}(\lambda) \stackrel{\text{def}}{=} \begin{cases} W_{\theta_l}(\lambda) = I_Y + \tilde{C}(\lambda - A)^{-1} B, \\ W_{\theta_r}(\lambda) = I_Y + C(\lambda - A_W)^{-1} \tilde{B}, \end{cases} \quad (7.26)$$

where  $|\operatorname{Im} \lambda| < \omega_{\theta}$ . Using (7.17) and (7.18) we can define the *weighting pattern*  $k_{\theta} : \mathbb{R} \times Y \rightarrow Y$  of an extended PS-realization by

$$k_{\theta}(t, y) \stackrel{\text{def}}{=} k_{\theta_l}(t, y) = k_{\theta_r}(t, y),$$

where  $t \in \mathbb{R}$  a.e. and  $y \in Y$ . Then for each  $\mu \in [0, \omega_{\theta})$  we have  $e^{\mu|\cdot|} k_{\theta}(\cdot, y) \in L^2(\mathbb{R}; Y)$  for each  $y \in Y$ . For each  $y \in Y$  we then have

$$[W_{\theta}(\lambda) - I_Y]y = \int_{-\infty}^{\infty} e^{i\lambda t} k_{\theta}(t, y) dt, \quad |\operatorname{Im} \lambda| < \omega_{\theta}.$$

Observe that we **cannot** write  $k_{\theta}(t, y) = -i\tilde{C}E(t; -iA)By$ , since  $\tilde{C}$  need not extend to a bounded linear operator from  $V$  into  $Y$ . Moreover, since  $\tilde{B}$  need not be a bounded linear operator from  $Y$  into  $W$ , we **cannot** write  $k_{\theta}(t)y$  with  $k_{\theta}(t) \in \mathcal{L}(Y)$  a.e. instead of  $k_{\theta}(t, y)$ . An example of an extended PS-realization  $\theta$  where  $k_{\theta}(\cdot, y)$  cannot be represented as an  $\mathcal{L}(Y)$ -valued function acting on  $y \in Y$ , has been given in [123, Sec. 4].

We now define the *input-output operator* as follows:

$$(T_{\theta}\phi)(t) = \int_{-\infty}^{\infty} k_{\theta}(t-s, \phi(s)) ds, \quad (7.27)$$

where the integral in (7.27) is a Pettis integral. Then for every  $\mu \in [0, \omega_\theta)$  we have, as a result of Lemma 2.11,

$$\left\| e^{\mu|\cdot|} T_\theta \phi \right\|_{L^2(\mathbb{R}; Y)} \leq \text{const.} \|e^{\mu|\cdot|} \phi\|_{L^2(\mathbb{R}; Y)}.$$

Thus

$$T_\theta = T_{\theta_t} = T_{\theta_r}.$$

Hence, the input-output operator  $T_\theta$  is bounded on  $L^2(\mathbb{R}; Y)$  with norm

$$\|T_\theta\|_{L^2(\mathbb{R}; Y)} = \sup_{\lambda \in \mathbb{R}} \|W_\theta(\lambda) - I_Y\|. \quad (7.28)$$

## 7.2.2 Duality of extended PS-realizations

The dual of a Pritchard-Salamon system has been defined in [159] using the natural imbedding of  $W$  into  $V$  instead of an arbitrary continuous and dense imbedding  $\tau : W \rightarrow V$ , thus complicating the construction. In [96] a more transparent definition of the dual has been given, where  $\tau^* : W \rightarrow V$  takes the place of  $\tau$ . Here we present the definition given in [97], but give a more illuminating account of the construction. In fact, we also prove that the natural dual of a right PS-realization is a left PS-realization. At the end of this subsection we discuss the difficulties encountered in constructing the natural dual of a left PS-realization.

**Theorem 7.10.** *Let  $\theta_r = (A_W, \tilde{B}, C; W; Y)$  be a right PS-realization. Then*

$$\theta_r^* \stackrel{\text{def}}{=} (A_W^*, C^*, \tilde{B}^{(*)}; W; Y), \quad (7.29)$$

where, for  $|\text{Im } \lambda| < \omega_{\theta_r}$ ,

$$\mathcal{D}(\tilde{B}^{(*)}) = \mathcal{D}(A_W^*), \quad \tilde{B}^{(*)} \stackrel{\text{def}}{=} [(\bar{\lambda} - A_W)^{-1} \tilde{B}]^* (\lambda - A_W^*), \quad (7.30)$$

is a left PS-realization satisfying

$$W_{\theta_r^*}(\lambda) = W_{\theta_r}(\bar{\lambda})^*, \quad |\text{Im } \lambda| < \omega_{\theta_r}, \quad (7.31a)$$

$$\langle k_{\theta_r^*}(t, y), z \rangle = \langle y, k_{\theta_r}(-t, z) \rangle, \quad y, z \in Y, \quad t \in \mathbb{R} \text{ a.e.} \quad (7.31b)$$

Moreover, if  $\theta = (A, B, C; V, W, \tau; Y)$  is an extended PS-realization, then

$$\theta^* \stackrel{\text{def}}{=} (A_W^*, C^*, B^*; W, V, \tau^*; Y) \quad (7.32)$$

is an extended Pritchard-Salamon realization and  $(\theta^*)^* = \theta$ .

Note that it does **not** follow from the closedness of  $\lambda - A_W$  and the boundedness of  $[(\bar{\lambda} - A_W)^{-1} \tilde{B}]^*$  that  $\tilde{B}^{(*)}$  is a closed operator.

*Proof.* Let  $\theta_r$  in (7.29) be a right PS-realization. Then  $-iA_W^* = -(-iA_W)^*$  is exponentially dichotomous and  $C^* \in \mathcal{L}(Y, W)$ . Further, the operator  $\tilde{B}^{(*)}$  given by (7.30) is well defined and its domain coincides with that of  $A_W^*$ . Moreover, from (7.15) we see that  $\Lambda_{\theta_r^*}$  defined by

$$(\Lambda_{\theta_r^*} w)(t) = \tilde{B}^{(*)} E(t; -iA_W^*) w = F(t, w), \quad w \in \mathcal{D}(A_W^*),$$

extends to a bounded linear operator from  $W$  into  $L^2(\mathbb{R}; Y)$ . Consequently,  $\theta_r^*$  is a left PS-realization. Its transfer function is given by

$$\begin{aligned} W_{\theta_r^*}(\lambda) &= I_Y + \tilde{B}^{(*)}(\lambda - A_W^*)^{-1} C^* \\ &= I_Y + [(\bar{\lambda} - A_W)^{-1} \tilde{B}]^* (\lambda - A_W^*)(\lambda - A_W^*)^{-1} C^* \\ &= I_Y + [(\bar{\lambda} - A_W)^{-1} \tilde{B}]^* C^* \\ &= \left[ I_Y + C(\bar{\lambda} - A_W)^{-1} \tilde{B} \right]^* = W_{\theta_r}(\bar{\lambda})^*. \end{aligned}$$

For  $y, z \in Y$  we then have

$$\langle [W_{\theta_r^*}(\lambda) - I_Y] y, z \rangle = \langle y, [W_{\theta_r}(\bar{\lambda}) - I_Y] z \rangle,$$

and therefore

$$\int_{-\infty}^{\infty} e^{i\lambda t} \langle k_{\theta_r^*}(t, y), z \rangle dt = \int_{-\infty}^{\infty} e^{-i\lambda t} \langle y, k_{\theta_r}(t, z) \rangle dt,$$

which implies (7.31b).

Let us now assume that  $\theta = (A, B, C; V, W, \tau; Y)$  is an extended PS-realization and define  $\theta^*$  by (7.32). Then, by the above,  $\theta_r^*$  is a left PS-realization. Also,  $-iA^*$  and  $-iA_W^*$  are exponentially dichotomous,  $B^* \in \mathcal{L}(V, Y)$ , and  $C^* \in \mathcal{L}(Y, W)$ . Since  $A_W$  is the part of  $A$  in  $W$  (with respect to  $\tau$ ), we have (7.19) for all  $\lambda$  satisfying  $|\operatorname{Im} \lambda| < \omega_\theta$ . Taking the adjoint we get

$$(\lambda - iA_W^*)^{-1} \tau^* = \tau^* (\lambda - iA^*)^{-1}, \quad |\operatorname{Im} \lambda| < \omega_\theta,$$

which implies that  $A^*$  is the part of  $A_W^*$  in  $V$  (with respect to  $\tau^*$ ). Using (7.19) and its implication  $E(t; -iA)\tau = \tau E(t; -iA_W)$  for  $0 \neq t \in \mathbb{R}$  and for  $t = 0^\pm$ , we get

$$\tau^* E(t; -iA^*) = E(t; -iA_W^*) \tau^*, \quad 0 \neq t \in \mathbb{R} \text{ or } t = 0^\pm.$$

The remaining two conditions follow from the results of the following calculations (in which  $\phi^\#(t) = \phi(-t)$ ):

$$\begin{aligned} \langle \Lambda_{\theta^*} \tau^* x, \phi \rangle_{L^2(\mathbb{R}; Y)} &= \int_{-\infty}^{\infty} \langle (\Lambda_{\theta^*} \tau^* x)(t), \phi(t) \rangle_Y dt \\ &= \int_{-\infty}^{\infty} \langle B^* E(t; -iA^*) x, \phi(t) \rangle_Y dt \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \langle x, E(-t; -iA)B\phi(t) \rangle_V dt \\
&= \int_{-\infty}^{\infty} \langle x, E(t; -iA)B\phi(-t) \rangle_V dt \\
&= \langle x, \tau \Gamma_{\theta} \phi^{\#} \rangle_V \\
&= \langle \tau^* x, \Gamma_{\theta} \phi^{\#} \rangle_W,
\end{aligned}$$

so that

$$\|\Lambda_{\theta^*} \tau^* x\|_{L^2(\mathbb{R}; Y)} \leq \|\tau^* x\|_W \sup_{\|\phi\|_{L^2(\mathbb{R}; Y)}=1} \|\Gamma_{\theta} \phi^{\#}\|_W = \|\Gamma_{\theta}\| \|\tau^* x\|_W,$$

and

$$\begin{aligned}
\langle \Gamma_{\theta^*} \phi, \tau x \rangle_V &= \langle \tau^* \Gamma_{\theta^*} \phi, x \rangle_W \\
&= \int_{-\infty}^{\infty} \langle E(t; -iA_W^*) C^* \phi(t), x \rangle_W dt \\
&= \int_{-\infty}^{\infty} \langle E(-t; -iA_W)^* C^* \phi(t), x \rangle_W dt \\
&= \int_{-\infty}^{\infty} \langle \phi(-t), CE(t; -iA_W) x \rangle_Y dt \\
&= \langle \phi^{\#}, \Lambda_{\theta} \tau x \rangle_{L^2(\mathbb{R}; Y)},
\end{aligned}$$

so that

$$\|\Gamma_{\theta^*} \phi\|_V \leq \|\phi\|_{L^2(\mathbb{R}; Y)} \sup_{\|\tau x\|_V=1} \|\Lambda_{\theta} \tau x\|_{L^2(\mathbb{R}; Y)} = \|\Lambda_{\theta}\| \|\phi\|_{L^2(\mathbb{R}; Y)}.$$

Thus  $\theta^*$  is an extended PS-realization, as claimed. We have in fact established the equalities

$$\Lambda_{\theta^*} = (\Gamma_{\theta} \mathcal{J}_Y)^*, \quad \Gamma_{\theta^*} = (\mathcal{J}_Y \Lambda_{\theta})^*,$$

where  $\mathcal{J}_Y$  is the unitary operator on  $L^2(\mathbb{R}, Y)$  given by  $\mathcal{J}_Y \phi = \phi^{\#}$ .

Finally, as in Subsection 7.1.1, we prove that  $(\theta^*)^* = \theta$ . □

Let  $\theta_l = (A, B, \tilde{C}; V; Y)$  be a left PS-realization. To prove that

$$\theta_l^* = (A^*, \tilde{C}^*, B^*; V; Y)$$

is a right PS-realization, we need to define the putative operator

$$\tilde{C}^* \stackrel{\text{def}}{=} (\lambda - A^*) \left[ \tilde{C}(\bar{\lambda} - A)^{-1} \right]^*$$

on all of  $Y$ , i.e., we need to prove that, for each  $y \in Y$ ,

$$\left[ \tilde{C}(\bar{\lambda} - A)^{-1} \right]^* y \in \mathcal{D}(A^*).$$

Of course, we can define  $\tilde{C}^*$  as a closed operator on the domain of those vectors  $y \in Y$  for which  $\left[\tilde{C}(\bar{\lambda} - A)^{-1}\right]^* y \in \mathcal{D}(A^*)$ , but this is not sufficient to arrive at a right PS-realization. Now note that for  $|\operatorname{Im} \lambda| < \omega_{\theta_l}$ ,  $x \in \mathcal{D}(A)$ , and  $y \in Y$

$$\langle (\bar{\lambda} - A)x, \left[\tilde{C}(\bar{\lambda} - A)^{-1}\right]^* y \rangle = \langle \tilde{C}x, y \rangle.$$

In general, this relation cannot be extended (from  $x \in \mathcal{D}(A)$ ) to a bounded linear functional defined on all  $x \in V$ , because the possible unboundedness of  $\tilde{C}$  implies that the closed and densely defined linear operator  $\tilde{C}^*(Y \rightarrow V)$  need not be defined on all of  $Y$ . In other words, in general  $\theta_l^*$  is not a right PS-realization.

### 7.2.3 Generating extended PS-realizations

In this subsection we construct an extended PS-realization  $\theta^\times$  from a given extended PS-realization  $\theta$  such that

$$W_{\theta^\times}(\lambda) = W_\theta(\lambda)^{-1}, \quad \lambda \in \mathbb{R}.$$

We also construct the product  $\theta = \theta_1 \theta_2$  of two extended PS-realizations such that

$$W_\theta(\lambda) = W_{\theta_1}(\lambda)W_{\theta_2}(\lambda), \quad \lambda \in \mathbb{R}.$$

Finally, we study spectral factorizations of transfer functions of extended PS-realizations. Analogous results are derived for left PS-realizations and for right PS-realizations.

Let us start with the construction of  $\theta^\times$ . Since it requires  $W_\theta(\lambda)^{-1}$  to exist and to converge to  $I_Y$  in the strong operator topology as  $|\lambda| \rightarrow \infty$  within a strip about the real line, we need to strengthen our assumptions on  $\theta$ .

**Theorem 7.11.** *Let  $\theta_l = (A, B, \tilde{C}; V; Y)$  be a left PS-realization such that  $W_{\theta_l}(\lambda)^{-1}$  exists and is bounded on some strip  $|\operatorname{Im} \lambda| \leq \varepsilon$ , where  $\varepsilon \in (0, \omega_{\theta_l})$ . Put  $A^\times = A - B\tilde{C}$  with  $\mathcal{D}(A^\times) = \mathcal{D}(A)$ . Then*

$$\theta_l^\times = (A^\times, B, -\tilde{C}; V; Y)$$

*is a left PS-realization and*

$$W_{\theta_l^\times}(\lambda) = W_{\theta_l}(\lambda)^{-1}, \quad |\operatorname{Im} \lambda| \leq \varepsilon.$$

*Similarly, let  $\theta_r = (A_W, \tilde{B}, C; W; Y)$  be a right PS-realization such that  $W_{\theta_r}(\lambda)^{-1}$  exists and is bounded on some strip  $|\operatorname{Im} \lambda| \leq \varepsilon$ , where  $\varepsilon \in (0, \omega_{\theta_r})$ . Put  $A_W^\times = A_W - \tilde{B}C$  with  $\mathcal{D}(A_W^\times) = \mathcal{D}(A_W)$ . Then*

$$\theta_r^\times = (A_r^\times, \tilde{B}, -C; W; Y)$$

is a right PS-realization and

$$W_{\theta_r^\times}(\lambda) = W_{\theta_r}(\lambda)^{-1}, \quad |\operatorname{Im} \lambda| \leq \varepsilon.$$

Finally, let  $\theta = (A, B, C; V, W, \tau; Y)$  be an extended PS-realization such that  $W_\theta(\lambda)^{-1}$  exists and is bounded on some strip  $|\operatorname{Im} \lambda| \leq \varepsilon$ , where  $\varepsilon \in (0, \omega_\theta)$ . Put  $A^\times = A - B\tilde{C}$  with  $\mathcal{D}(A^\times) = \mathcal{D}(A)$ . Then

$$\theta^\times = (A^\times, B, -C; V, W, \tau; Y)$$

is an extended PS-realization and

$$W_{\theta^\times}(\lambda) = W_\theta(\lambda)^{-1}, \quad |\operatorname{Im} \lambda| \leq \varepsilon.$$

*Proof.* Let  $W(\lambda)$  be the transfer function of  $\theta_l$ ,  $\theta_r$ , or  $\theta$ , and write  $\omega$  for  $\omega_{\theta_l}$ ,  $\omega_{\theta_r}$ , or  $\omega_\theta$ , whatever the case may be. Then for any strongly measurable vector function  $\phi : \mathbb{R} \rightarrow Y$  such that  $e^{\varepsilon|\cdot|}\phi \in L^2(\mathbb{R}; Y)$  for any  $\varepsilon \in (0, \omega)$ , we have, for each  $\varepsilon \in (0, \mu)$ ,

$$W(\lambda)^{-1}\hat{\phi}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} H(t) dt, \quad |\operatorname{Im} \lambda| \leq \varepsilon,$$

where

$$H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} W(\lambda)^{-1}\hat{\phi}(\lambda) d\lambda$$

satisfies  $e^{\varepsilon|\cdot|}H \in L^2(\mathbb{R}; Y)$ . This is due to the invertibility condition on the transfer function  $W$ .

**1. Left PS-realization.** The strip  $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \leq \varepsilon\} \subset \rho(A^\times)$  and

$$(\lambda - A^\times)^{-1} = (\lambda - A)^{-1} - (\lambda - A)^{-1} B W_{\theta_l}(\lambda)^{-1} \tilde{C} (\lambda - A)^{-1}, \quad (7.33a)$$

$$\tilde{C} (\lambda - A^\times)^{-1} = W_{\theta_l}(\lambda)^{-1} \tilde{C} (\lambda - A)^{-1}, \quad (7.33b)$$

where  $|\operatorname{Im} \lambda| \leq \varepsilon$ . Since

$$\tilde{C} (\lambda - A)^{-1} x = -i \int_{-\infty}^{\infty} e^{i\lambda t} (\Lambda_{\theta_l} x)(t) dt, \quad |\operatorname{Im} \lambda| < \omega_{\theta_l}, \quad x \in V,$$

we get, from (7.33b) and Condition L3,

$$\tilde{C} (\lambda - A^\times)^{-1} x = -i \int_{-\infty}^{\infty} e^{i\lambda t} H_l(t; x) dt, \quad |\operatorname{Im} \lambda| \leq \varepsilon, \quad x \in V,$$

where  $e^{\varepsilon|\cdot|}H_l(\cdot; x) \in L^2(\mathbb{R}; Y)$ . Then for  $|\operatorname{Im} \lambda| \leq \varepsilon$  and  $x \in V$  we have, because of (7.33a),

$$(\lambda - A^\times)^{-1} x = -i \int_{-\infty}^{\infty} e^{i\lambda t} \left[ E(t; -iA)x - \int_{-\infty}^{\infty} E(t-s; -iA) B H_l(s; x) ds \right] dt,$$

where it is easily verified that the expression between square brackets multiplied by  $e^{\varepsilon|t|}$  belongs to  $L^\infty(\mathbb{R}; V)$ , uniformly in  $x$  on bounded subsets of  $V$ . According to Theorem 1.7,  $-iA^\times$  is exponentially dichotomous. Consequently,  $\theta_t^\times$  is a left PS-realization. We easily obtain that

$$W_{\theta_t^\times}(\lambda) = I_Y - \tilde{C}(\lambda - A^\times)^{-1}B = I_Y - W_{\theta_t}(\lambda)^{-1}\tilde{C}(\lambda - A)^{-1}B = W_{\theta_t}(\lambda)^{-1},$$

as claimed.

**2. Right PS-realization.** We derive in a similar way

$$(\lambda - A_W^\times)^{-1} = (\lambda - A_W)^{-1} - (\lambda - A_W)^{-1}\tilde{B}W_{\theta_r}(\lambda)^{-1}C(\lambda - A_W)^{-1}, \quad (7.34a)$$

$$(\lambda - A_W^\times)^{-1}\tilde{B} = (\lambda - A_W)^{-1}\tilde{B}W_{\theta_r}(\lambda)^{-1}, \quad (7.34b)$$

where  $|\operatorname{Im} \lambda| \leq \varepsilon$ . Now note that

$$C(\lambda - A_W)^{-1}x = -i \int_{-\infty}^{\infty} e^{i\lambda t} CE(t; -iA_W)x, \quad |\operatorname{Im} \lambda| < \omega_{\theta_r}, \quad x \in W,$$

where the vector function  $e^{\varepsilon|\cdot|}CE(\cdot; -iA_W)x \in L^2(\mathbb{R}; W)$  for some  $\varepsilon > 0$ . Since  $(\lambda - A_W)^{-1}\tilde{B}W_{\theta_r}(\lambda)^{-1}$  is bounded in a horizontal strip about the real line, we get

$$(\lambda - A_W^\times)^{-1}x = -i \int_{-\infty}^{\infty} e^{i\lambda t} [E(t; -iA_W)x + H(t; x)] dt, \quad (7.35)$$

where for some  $\varepsilon > 0$  the vector function  $e^{\varepsilon|\cdot|}H(\cdot; x) \in L^\infty(\mathbb{R}; W)$ . According to Theorem 1.7,  $-iA_W^\times$  is exponentially dichotomous. Also, the map  $\Gamma_{\theta_r^\times}$ , which on each Bochner integrable step function  $\phi: \mathbb{R} \rightarrow Y$  is given by

$$\Gamma_{\theta_r^\times} \phi = \int_{-\infty}^{\infty} E(s; -iA_W^\times)\tilde{B}\phi(s) ds,$$

extends to a bounded linear operator from  $L^2(\mathbb{R}; Y)$  into  $W$ . Indeed, using (7.35) we have

$$\Gamma_{\theta_r^\times} \phi - \Gamma_{\theta_r} \phi = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(t-s; CE(s; -iA_W)\tilde{B}\phi(t)) ds dt.$$

Letting  $\phi(t) = \sum_{j=1}^n \varphi_j(t)x_j$  for  $x_1, \dots, x_n \in Y$  and  $\varphi_1, \dots, \varphi_n \in L^2(\mathbb{R})$  with mutually disjoint support, we get, as a consequence of Lemma 2.11,

$$\begin{aligned} \|\Gamma_{\theta_r^\times} \phi - \Gamma_{\theta_r} \phi\| &\leq \text{const.} \sum_{j=1}^n \int_{-\infty}^{\infty} |\varphi_j(s)| \|CE(s; -iA_W)\tilde{B}x_j\| ds \\ &\leq \text{const.} \sum_{j=1}^n \|\varphi_j\|_2 \left[ \int_{-\infty}^{\infty} \|CE(s; -iA_W)\tilde{B}x_j\|^2 ds \right]^{1/2} \\ &\leq \text{const.}_2 \sum_{j=1}^n \|\varphi_j\|_2 \|x_j\| = \text{const.}_2 \|\phi\|_{L^2(\mathbb{R}; Y)}. \end{aligned}$$

Hence  $\theta_r^\times$  satisfies Condition R3 as well. Consequently,  $\theta_r^\times$  is a right PS-realization.

**3. Extended PS-realization.** Equations (7.33a) and (7.34a) imply that for  $|\operatorname{Im} \lambda| \leq \varepsilon$ ,

$$\begin{aligned} (\lambda - A^\times)^{-1} \tau &\stackrel{(7.33a)}{=} (\lambda - A)^{-1} \tau - (\lambda - A)^{-1} B W_{\theta_l}(\lambda)^{-1} \tilde{C} (\lambda - A)^{-1} \tau \\ &= \tau (\lambda - A_W)^{-1} - \tau (\lambda - A_W)^{-1} \tilde{B} W_{\theta_r}(\lambda)^{-1} C (\lambda - A_W)^{-1} \\ &\stackrel{(7.34a)}{=} \tau (\lambda - A_W^\times)^{-1}, \end{aligned}$$

where (7.19), (7.24), (7.25), and (7.26) are used at the second equality sign. Consequently,  $A_W^\times = (A^\times)_W$ , the part of  $A^\times$  in  $W$ . Also,

$$\tilde{C} (\lambda - A^\times)^{-1} \tau = C (\lambda - A_W^\times)^{-1}, \quad (\lambda - A^\times)^{-1} B = \tau (\lambda - A_W^\times)^{-1} \tilde{B},$$

implying that  $\tilde{C}$  and  $\tilde{B}$  are the operators defined in terms of  $\theta^\times$  in the sense of Theorem 7.9. We also get

$$W_\theta(\lambda)^{-1} = I_Y - \tilde{C} (\lambda - A^\times)^{-1} B = I_Y - C (\lambda - A_W^\times)^{-1} \tilde{B}, \quad |\operatorname{Im} \lambda| \leq \varepsilon,$$

which coincides with the transfer function of  $\theta^\times$ .

To prove that  $\theta^\times$  is an extended PS-realization, we rely on the fact that  $\theta_l^\times$  is a left PS-realization and  $\theta_r^\times$  is a right PS-realization. First observe that

$$\begin{aligned} \tilde{C} (\lambda - A^\times)^{-1} \tau &= \tilde{C} \left[ (\lambda - A)^{-1} - (\lambda - A)^{-1} B W_{\theta_l}(\lambda)^{-1} \tilde{C} (\lambda - A)^{-1} \right] \tau \\ &= C (\lambda - A_W)^{-1} - [W_{\theta_l}(\lambda) - I_Y] W_{\theta_l}(\lambda)^{-1} C (\lambda - A_W)^{-1} \\ &= W_{\theta_l}(\lambda)^{-1} C (\lambda - A_W)^{-1} = W_{\theta_r}(\lambda)^{-1} C (\lambda - A_W)^{-1} \\ &= [I_Y - C (\lambda - A_W^\times)^{-1} \tilde{B}] C (\lambda - A_W)^{-1} = C (\lambda - A_W^\times)^{-1}. \end{aligned}$$

Then the coupling relation

$$\tilde{C} E(t; -iA^\times) \tau = C E(t; -iA_W^\times), \quad 0 \neq t \in \mathbb{R},$$

which follows by inverse Laplace transformation, implies that for  $x \in W$ ,

$$\|C E(\cdot; -iA_W^\times) x\|_{L^2(\mathbb{R}; Y)} = \|\tilde{C} E(\cdot; -iA^\times) \tau x\|_{L^2(\mathbb{R}; Y)} \leq \|\Lambda_{\theta_l^\times}\| \|\tau x\|_V,$$

so that  $\theta^\times$  satisfies Condition E4. In the same way we prove the coupling relation

$$\tau E(t; -iA_W^\times) \tilde{B} = E(t; -iA^\times) B$$

from the equality

$$\tau (\lambda - A_W^\times)^{-1} \tilde{B} = (\lambda - A^\times)^{-1} B.$$

As a result, for  $\phi \in L^2(\mathbb{R}; Y)$  of the form  $\phi(t) = \sum_{j=1}^n \varphi_j(t) x_j$  with scalar functions  $\varphi_1, \dots, \varphi_n \in L^2(\mathbb{R})$  and vectors  $x_1, \dots, x_n \in Y$  we obtain

$$\Gamma_{\theta^\times} \phi = \int_{-\infty}^{\infty} E(s; -iA^\times) B \phi(s) ds = \int_{-\infty}^{\infty} \tau E(s; -iA_W^\times) B \phi(s) ds = \tau \Gamma_{\theta_r^\times} \phi,$$



implying that  $\Gamma_{\theta^\times} \phi \in \tau[W]$  and

$$\|\tau^{-1}\Gamma_{\theta^\times} \phi\|_W \leq \|\Gamma_{\theta_r^\times}\| \|\phi\|_{L^2(\mathbb{R}; Y)}.$$

Thus  $\theta^\times$  satisfies Condition E5. Consequently,  $\theta^\times$  is an extended PS-realization.  $\square$

Let us now derive the product rule for extended PS-realizations, left PS-realizations, and right PS-realizations.

**Theorem 7.12.** *For  $j = 1, 2$ , let  $\theta_{lj} = (A_j, B_j, \tilde{C}_j; V_j; Y)$  be two left PS-realizations. Put  $\theta_l = (A, B, \tilde{C}; V; Y)$ , where*

$$V = V_1 \dot{+} V_2, \quad \mathcal{D}(A) = \mathcal{D}(A_1) \dot{+} \mathcal{D}(A_2), \quad (7.36a)$$

$$A = \begin{pmatrix} A_1 & B_1 \tilde{C}_2 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \tilde{C} = (\tilde{C}_1 \quad \tilde{C}_2). \quad (7.36b)$$

Then  $\theta_l$  is a left PS-realization and

$$W_{\theta_l}(\lambda) = W_{\theta_{l1}}(\lambda)W_{\theta_{l2}}(\lambda), \quad |\operatorname{Im} \lambda| < \min(\omega_{\theta_{l1}}, \omega_{\theta_{l2}}). \quad (7.37)$$

For  $j = 1, 2$ , let  $\theta_{rj} = (A_{j,W}, \tilde{B}_j, C_j; W_j; Y)$  be two right PS-realizations. Put  $\theta_r = (A_W, \tilde{B}, C; W; Y)$ , where

$$W = W_1 \dot{+} W_2, \quad \mathcal{D}(A_W) = \mathcal{D}(A_{1,W}) \dot{+} \mathcal{D}(A_{2,W}), \quad (7.38a)$$

$$A_W = \begin{pmatrix} A_{1,W} & \tilde{B}_1 C_2 \\ 0 & A_{2,W} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}, \quad C = (C_1 \quad C_2). \quad (7.38b)$$

Then  $\theta_r$  is a right PS-realization and

$$W_{\theta_r}(\lambda) = W_{\theta_{r1}}(\lambda)W_{\theta_{r2}}(\lambda), \quad |\operatorname{Im} \lambda| < \min(\omega_{\theta_{r1}}, \omega_{\theta_{r2}}).$$

Finally, for  $j = 1, 2$ , let  $\theta_j = (A_j, B_j, C_j; V_j, W_j, \tau_j; Y)$  be two extended PS-realizations. Put  $\theta = (A, B, C; V, W, \tau; Y)$ , where  $A, B, C, V$ , and  $W$  are given by (7.36) and (7.38) and  $\tau = \tau_1 \dot{+} \tau_2$ . Then  $\theta$  is an extended PS-realization and

$$W_\theta(\lambda) = W_{\theta_1}(\lambda)W_{\theta_2}(\lambda), \quad |\operatorname{Im} \lambda| < \min(\omega_{\theta_1}, \omega_{\theta_2}).$$

If  $W_{\theta_1}(\lambda)^{-1}$  and  $W_{\theta_2}(\lambda)^{-1}$  are bounded in the strip  $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \leq \varepsilon\}$  for some  $\varepsilon \in (0, \min(\omega_{\theta_1}, \omega_{\theta_2}))$ , then

$$\theta^\times = (\theta_2)^\times (\theta_1)^\times,$$

where

$$A^\times = \begin{pmatrix} A_1^\times & 0 \\ -B_2^\times \tilde{C}_1 & A_2^\times \end{pmatrix}, \quad A_W^\times = \begin{pmatrix} A_{1,W}^\times & 0 \\ -\tilde{B}_2^\times C_1 & A_{2,W}^\times \end{pmatrix}.$$

This result holds for left, right, and extended PS-realizations, whatever may be the situation.

*Proof of Theorem 7.12.* It easily follows that, for  $|\operatorname{Im} \lambda| < \min(\omega_{\theta_1}, \omega_{\theta_2})$ ,

$$(\lambda - A)^{-1} = \begin{pmatrix} (\lambda - A_1)^{-1} & (\lambda - A_1)^{-1} B_1 \tilde{C}_2 (\lambda - A_2)^{-1} \\ 0 & (\lambda - A_2)^{-1} \end{pmatrix}$$

for  $|\operatorname{Im} \lambda| < \min(\omega_{\theta_{l1}}, \omega_{\theta_{l2}})$  and

$$(\lambda - A_W)^{-1} = \begin{pmatrix} (\lambda - A_{1,W})^{-1} & (\lambda - A_{1,W})^{-1} \tilde{B}_1 C_2 (\lambda - A_{2,W})^{-1} \\ 0 & (\lambda - A_{2,W})^{-1} \end{pmatrix}$$

for  $|\operatorname{Im} \lambda| < \min(\omega_{\theta_{r1}}, \omega_{\theta_{r2}})$ . We then have

$$\begin{aligned} (\lambda - A)^{-1} x &= -i \int_{-\infty}^{\infty} e^{i\lambda t} E(t, x) dt, & x \in V, \\ (\lambda - A_W)^{-1} x &= -i \int_{-\infty}^{\infty} e^{i\lambda t} E_W(t, x) dt, & x \in W, \end{aligned}$$

where for  $x = (x_1, x_2)$ ,

$$\begin{aligned} E(t, x) &= \begin{pmatrix} E(t; -iA_1)x_1 & \int_{-\infty}^{\infty} E(t-s; -iA_1)B_1(\Lambda_{\theta_{l2}}x_2)(s) ds \\ 0 & E(t; -iA_2)x_2 \end{pmatrix}, \\ E_W(t, x) &= \begin{pmatrix} E(t; -iA_{1,W})x_1 & [E_W(t, x)]_{12} \\ 0 & E(t; -iA_{2,W})x_2 \end{pmatrix}, \\ [E_W(t, x)]_{12} &= \int_{-\infty}^{\infty} E(t-s; -iA_{1,W})\tilde{B}_1 C_2 E(s; -iA_{2,W})x_2 ds. \end{aligned}$$

Now note that  $\theta_{l2}$  satisfies Condition L3. It is then easily seen that, for  $0 \leq \mu < \min(\omega_{\theta_{l1}}, \omega_{\theta_{l2}})$ ,

$$\|E(t, x)\| \leq \text{const.} e^{-\mu|t|} \|x\|_V, \quad x \in V.$$

As a result of Theorem 1.7,  $-iA$  is exponentially dichotomous. Next, using that  $e^{\mu|\cdot|} E(\cdot; -iA_{1,W})\tilde{B}_1 y \in L^2(\mathbb{R}; W)$  for  $y \in Y$  whenever  $0 \leq \mu < \min(\omega_{\theta_{r1}}, \omega_{\theta_{r2}})$  and  $\theta_{r1}$  is a right PS-realization, it is easily seen that for  $0 \leq \mu < \min(\omega_{\theta_{r1}}, \omega_{\theta_{r2}})$

$$\|E_W(t, x)\| \leq \text{const.} e^{-\mu|t|} \|x\|_W, \quad x \in W.$$

As a result of Theorem 1.7,  $-iA_W$  is exponentially dichotomous.

For the above ranges of  $\lambda$  we have

$$\tilde{C}(\lambda - A)^{-1} = (\tilde{C}_1(\lambda - A_1)^{-1} \quad W_{\theta_{l1}}(\lambda) \tilde{C}_2(\lambda - A_2)^{-1}), \quad (7.39a)$$

$$(\lambda - A_W)^{-1} \tilde{B} = \begin{pmatrix} (\lambda - A_{1,W})^{-1} \tilde{B}_1 W_{\theta_{r2}}(\lambda) \\ (\lambda - A_{2,W})^{-1} \tilde{B}_2 \end{pmatrix}. \quad (7.39b)$$

We then get by inverse Laplace transformation

$$\Lambda_{\theta_l} = (\Lambda_{\theta_{l1}} \quad [I + T_{\theta_{l1}}] \Lambda_{\theta_{l2}}), \quad \Gamma_{\theta_r} = \begin{pmatrix} \Gamma_{\theta_{r1}} [I + T_{\theta_{r2}}] \\ \Gamma_{\theta_{r2}} \end{pmatrix},$$

where  $T_{\theta_{l1}}$  and  $T_{\theta_{r2}}$  are the input-output operators of  $\theta_{l1}$  and  $\theta_{r2}$ . Since these input-output operators are bounded on  $L^2(\mathbb{R}; Y)$ , we see that the operators  $\Lambda_{\theta_l} : V \rightarrow L^2(\mathbb{R}; Y)$  and  $\Gamma_{\theta_r} : L^2(\mathbb{R}; Y) \rightarrow W$  are bounded. Thus  $\theta_l$  satisfies Condition L3 and  $\theta_r$  satisfies Condition R3. Consequently,  $\theta_l$  is a left PS-realization and  $\theta_r$  is a right PS-realization.

Let us now compute the transfer functions of  $\theta_l$  and  $\theta_r$ . Using (7.39a) we have

$$\begin{aligned} W_{\theta_l}(\lambda) &= I_Y + \tilde{C}(\lambda - A)^{-1}B \\ &= I_Y + \tilde{C}_1(\lambda - A_1)^{-1}B_1 + W_{\theta_{l1}}(\lambda)\tilde{C}_2(\lambda - A_2)^{-1}B_2 \\ &= W_{\theta_{l1}}(\lambda)W_{\theta_{l2}}(\lambda). \end{aligned}$$

In a similar way we have, with the help of (7.39b),

$$\begin{aligned} W_{\theta_r}(\lambda) &= I_Y + C(\lambda - A_W)^{-1}\tilde{B} \\ &= I_Y + C_1(\lambda - A_{1,W})^{-1}\tilde{B}_1W_{\theta_{r2}}(\lambda) + C_2(\lambda - A_{2,W})^{-1}\tilde{B}_2 \\ &= W_{\theta_{r1}}(\lambda)W_{\theta_{r2}}(\lambda). \end{aligned}$$

Let us now depart from the extended PS-realizations  $\theta_1$  and  $\theta_2$  and construct the compound

$$\theta = (A, B, C; V, W, \tau; Y),$$

where  $A, B, C, V, W$ , and  $\tau$  are given by (7.36) and (7.38) and  $\tau = \tau_1 \dot{+} \tau_2$ . Then the corresponding  $\theta_l = (A, B, \tilde{C}; V; Y)$  is a left PS-realization and the corresponding  $\theta_r = (A_W, \tilde{B}, C; W; Y)$  is a right PS-realization. Further,

$$\begin{aligned} (\lambda - A)^{-1}\tau &= \begin{pmatrix} (\lambda - A_1)^{-1}\tau_1 & (\lambda - A_1)^{-1}B_1\tilde{C}_2(\lambda - A_2)^{-1}\tau_2 \\ 0 & (\lambda - A_2)^{-1}\tau_2 \end{pmatrix} \\ &= \begin{pmatrix} \tau_1(\lambda - A_{1,W})^{-1} & \tau_1(\lambda - A_{1,W})^{-1}\tilde{B}_1C_2(\lambda - A_{2,W})^{-1} \\ 0 & \tau_2(\lambda - A_{2,W})^{-1} \end{pmatrix} \\ &= \tau(\lambda - A_W)^{-1}, \end{aligned}$$

so that  $A_W$  is the part of  $A$  in  $W$ . Using (7.39a) we compute

$$\begin{aligned} \tilde{C}(\lambda - A)^{-1}\tau &= (\tilde{C}_1(\lambda - A_1)^{-1}\tau_1 \quad W_{\theta_{l1}}(\lambda)\tilde{C}_2(\lambda - A_2)^{-1}\tau_2) \\ &= (C_1(\lambda - A_{1,W})^{-1} \quad W_{\theta_{l1}}(\lambda)C_2(\lambda - A_{2,W})^{-1}) \\ &= C(\lambda - A_W)^{-1}. \end{aligned}$$

Analogously, with the help of (7.39b) we get

$$\begin{aligned} \tau(\lambda - A_W)^{-1}\tilde{B} &= \begin{pmatrix} \tau_1(\lambda - A_{1,W})^{-1}\tilde{B}_1W_{\theta_{r2}}(\lambda) \\ \tau_2(\lambda - A_{2,W})^{-1}\tilde{B}_2 \end{pmatrix} \\ &= \begin{pmatrix} (\lambda - A_1)^{-1}B_1W_{\theta_2}(\lambda) \\ (\lambda - A_2)^{-1}B_2 \end{pmatrix} = (\lambda - A)^{-1}B. \end{aligned}$$

Consequently,  $\theta$  is an extended PS-realization, as claimed.  $\square$

### 7.2.4 Factorizing extended PS-realizations

Let us now present the analog of Theorem 7.4 for transfer functions of extended PS-realizations [97]. Let  $\theta = (A, B, C; V, W, \tau; Y)$  be an extended PS-realization such that  $W_\theta(\lambda)^{-1}$  is bounded in the strip  $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \leq \varepsilon\}$  for some  $\varepsilon \in (0, \omega_\theta)$ . Then we define  $V_\pm$ ,  $W_\pm$ ,  $V_\pm^\times$ , and  $W_\pm^\times$  as follows:

$$\begin{aligned} V_\pm &= \operatorname{Im} E(0^\pm; -iA), & V_\pm^\times &= \operatorname{Im} E(0^\pm; -iA^\times), \\ W_\pm &= \operatorname{Im} E(0^\pm; -iA_W), & W_\pm^\times &= \operatorname{Im} E(0^\pm; -iA_W^\times), \end{aligned}$$

where  $-E(0^-; -iA)$ ,  $-E(0^-; -iA^\times)$ ,  $-E(0^-; -iA_W)$ , and  $-E(0^-; -iA_W^\times)$  are the separating projections of the bisemigroups generated by  $-iA$ ,  $-iA^\times$ ,  $-iA_W$ , and  $-iA_W^\times$ , respectively.

**Theorem 7.13.** *Let  $\theta = (A, B, C; V, W, \tau; Y)$  be an extended PS-realization such that  $W_\theta(\lambda)^{-1}$  is bounded in the strip  $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \leq \varepsilon\}$  for some  $\varepsilon \in (0, \omega_\theta)$ . Then the following statements are equivalent:*

1.  $W_\theta(\cdot)$  has a left quasi-canonical factorization.
2. We have the decompositions  $V = V_- \dot{+} V_+^\times$  and  $W = W_- \dot{+} W_+^\times$ .
3. For every  $g \in L^2(\mathbb{R}^+; Y)$  the convolution equation

$$\phi(t) + \int_0^\infty k_\theta(t-s, \phi(s)) ds = g(t), \quad t \in \mathbb{R}^+, \quad (7.40)$$

is uniquely solvable in  $L^2(\mathbb{R}^+; Y)$ .

Moreover, we have the equivalent statements:

- 1'.  $W_\theta(\cdot)$  has a right quasi-canonical factorization.
- 2'. We have the decompositions  $V = V_+ \dot{+} V_-^\times$  and  $W = W_+ \dot{+} W_-^\times$ .
- 3'. For every  $g \in L^2(\mathbb{R}^-; Y)$  the convolution equation

$$\phi(t) + \int_{-\infty}^0 k_\theta(t-s, \phi(s)) ds = g(t), \quad t \in \mathbb{R}^-, \quad (7.41)$$

is uniquely solvable in  $L^2(\mathbb{R}^-; Y)$ .

In (7.40)–(7.41) the integrals are to be understood as Pettis integrals.

We restrict ourselves to proving the equivalence of conditions 1–3. The proof of the implication **(1)**  $\implies$  **(3)** proceeds almost exactly as in the proof of Theorem 7.4 and is omitted.

*Proof of Theorem 7.13.* **(2)**  $\implies$  **(1)** Let  $\Pi$  stand for the projection of  $V$  onto  $V_+^\times$  along  $V_-$  and let

$$\theta_l = (A, B, \tilde{C}; V; Y) \quad (7.42)$$

be a left PS-realization such that  $V = V_- \dot{+} V_+^\times$  and  $W_{\theta_l}(\lambda)^{-1}$  is bounded on  $|\operatorname{Im} \lambda| \leq \varepsilon$ . Suppose  $A_1, A_2, A_1^\times$ , and  $A_2^\times$  are the parts of  $A$  in  $V_-$ , of  $A$  in  $V_+^\times$ , of  $A^\times$  in  $V_-$ , and of  $A^\times$  in  $V_+^\times$ , respectively, so that  $A_1$  is the restriction of  $A$  to  $V_-$  and  $A_2^\times$  is the restriction of  $A^\times$  to  $V_+^\times$ . Let  $\tilde{C}_1$  and  $\tilde{C}_2$  be the restrictions of  $\tilde{C}$  to  $V_-$  and  $V_+^\times$ , and let  $B_1 \in \mathcal{L}(Y, V_-)$  and  $B_2 \in \mathcal{L}(Y, V_+^\times)$  have the same actions as  $(I_V - \Pi)B$  and  $\Pi B$ . Put

$$\begin{aligned}\theta_{l1} &= (A_1, B_1, \tilde{C}_1; V_-; Y), & \theta_{l2} &= (A_2, B_2, \tilde{C}_2; V_+^\times; Y), \\ \theta_{l1}^\times &= (A_1^\times, B_1, -\tilde{C}_1; V_-; Y), & \theta_{l2}^\times &= (A_2^\times, B_2, -\tilde{C}_2; V_+^\times; Y).\end{aligned}$$

Then for  $x \in V_-$  we have

$$\tilde{C}_1(\lambda - A_1)^{-1}x = \tilde{C}(\lambda - A)^{-1}x = -i \int_{-\infty}^{\infty} e^{i\lambda t} (\Lambda_{\theta_l} x)(t) dt,$$

while for  $x \in V_+^\times$  we have

$$\tilde{C}_2(\lambda - A_2^\times)^{-1}x = \tilde{C}(\lambda - A^\times)^{-1}x = i \int_{-\infty}^{\infty} e^{i\lambda t} (\Lambda_{\theta_l^\times} x)(t) dt.$$

Thus

$$\Lambda_{\theta_{l1}} = \Lambda_{\theta_l}|_{V_-} : V_- \rightarrow L^2(\mathbb{R}; Y) \quad \text{and} \quad \Lambda_{\theta_{l2}^\times} = \Lambda_{\theta_l^\times}|_{V_+^\times} : V_+^\times \rightarrow L^2(\mathbb{R}; Y)$$

are bounded. Furthermore,  $-iA_1$  and  $-iA_2^\times$  are restrictions of exponentially dichotomous operators and hence exponentially dichotomous. Therefore,  $\theta_{l1}$  and  $\theta_{l2}^\times$  are left PS-realizations. Moreover,

$$\begin{aligned}W_{\theta_l}(\lambda)W_{\theta_{l2}^\times}(\lambda) &= \left[ I_Y + \tilde{C}(\lambda - A)^{-1}B \right] \left[ I_Y - \tilde{C}(\lambda - A)^{-1}\Pi B \right] \\ &= I_Y + \tilde{C}(\lambda - A)^{-1}(I_V - \Pi)B = W_{\theta_{l1}}(\lambda).\end{aligned}\tag{7.43}$$

If  $\theta_l$  in (7.42) is a left PS-realization, then it does not follow immediately that  $\theta_{l2}$  and  $\theta_{l1}^\times$  are left PS-realizations too.

Now note that

$$(\lambda - A_2)^{-1}x = \Pi(\lambda - A)^{-1}x = -i \int_{-\infty}^{\infty} e^{i\lambda t} \Pi E(t; -iA)x dt$$

for  $x \in V_-$ , and

$$(\lambda - A_1^\times)^{-1} = (I_V - \Pi)(\lambda - A^\times)^{-1}x = -i \int_{-\infty}^{\infty} e^{i\lambda t} (I_V - \Pi)E(t; -iA^\times)x dt$$

for  $x \in V_+^\times$ , where  $\Pi E(t; -iA)x$  and  $(I_V - \Pi)E(t; -iA^\times)x$  are bounded above by  $M e^{-c|t|}$  for certain  $c, M > 0$ . It then follows from Theorem 1.7 that  $-iA_2$  and  $-iA_1^\times$  are exponentially dichotomous.

Next, let  $\Pi_W$  denote the projection of  $W$  onto  $W_+^\times$  along  $W_-$  and let

$$\theta_r = (A_W, \tilde{B}, C; W; Y) \quad (7.44)$$

be a right PS-realization such that  $W = W_- \dot{+} W_+^\times$  and  $W_{\theta_r}(\lambda)^{-1}$  is bounded on  $|\operatorname{Im} \lambda| \leq \varepsilon$ . Suppose  $A_{1,W}$ ,  $A_{2,W}$ ,  $A_{1,W}^\times$ , and  $A_{2,W}^\times$  are the parts of  $A_W$  in  $W_-$ , of  $A_W$  in  $W_+^\times$ , of  $A_W^\times$  in  $W_-$ , and of  $A_W^\times$  in  $W_+^\times$ , respectively, so that  $A_{1,W}$  is the restriction of  $A_W$  to  $W_-$  and  $A_{2,W}^\times$  is the restriction of  $A_W^\times$  to  $W_+^\times$ . Let  $C_1$  and  $C_2$  be the restrictions of  $C$  to  $W_-$  and  $W_+^\times$ , and let  $\tilde{B}_1 : Y \rightarrow W_-$  and  $\tilde{B}_2 : Y \rightarrow W_+^\times$  have the same actions as  $(I_W - \Pi_W)\tilde{B}$  and  $\Pi_W\tilde{B}$ . Put

$$\begin{aligned} \theta_{r1} &= (A_{1,W}, \tilde{B}_1, C_1; W_-; Y), & \theta_{r2} &= (A_{2,W}, \tilde{B}_2, C_2; W_+^\times; Y), \\ \theta_{r1}^\times &= (A_{1,W}^\times, \tilde{B}_1, -C_1; W_-; Y), & \theta_{r2}^\times &= (A_{2,W}^\times, \tilde{B}_2, -C_2; W_+^\times; Y). \end{aligned}$$

Then for  $\phi \in L^2(\mathbb{R}; Y)$  a step function the expressions

$$\Gamma_{\theta_{r2}} \phi = \int_{-\infty}^{\infty} E(s; -iA_{W,2}) \tilde{B}_2 \phi(s) ds = \Pi_W \Gamma_{\theta_r} \phi$$

and

$$\Gamma_{\theta_{r1}^\times} \phi = \int_{-\infty}^{\infty} E(s; -iA_{W,1}^\times) \tilde{B}_1 \phi(s) ds = (I_W - \Pi_W) \Gamma_{\theta_r^\times} \phi$$

extend to bounded linear operators from  $L^2(\mathbb{R}; Y)$  into  $W_+^\times$  and  $W_-$ , respectively. Also, for  $x \in W_+^\times$  we have

$$(\lambda - A_{W,2})^{-1}x = \Pi_W(\lambda - A_W)^{-1}x = -i \int_{-\infty}^{\infty} e^{i\lambda t} \Pi_W E(t; -iA_W) x dt,$$

so that  $-iA_{W,2}$  is exponentially dichotomous as a result of Theorem 1.7 and the exponential bound on  $\Pi_W E(t; -iA_W)$ . Similarly, for  $x \in W_-$  we have

$$\begin{aligned} (\lambda - A_{W,1}^\times)^{-1}x &= (I_W - \Pi_W)(\lambda - A_W^\times)^{-1}x \\ &= -i \int_{-\infty}^{\infty} e^{i\lambda t} (I_W - \Pi_W) E(t; -iA_W^\times) x dt, \end{aligned}$$

so that  $-iA_{W,1}^\times$  is exponentially dichotomous as a result of Theorem 1.7 and the exponential bound on  $(I_W - \Pi_W)E(t; -iA_W^\times)$ . Consequently,  $\theta_{r2}$  and  $\theta_{r1}^\times$  are right PS-realizations. Moreover,

$$\begin{aligned} W_{\theta_{r1}^\times}(\lambda) W_{\theta_r}(\lambda) &= \left[ I_Y - C(I - \Pi_W)(\lambda - A_W^\times)^{-1} \tilde{B} \right] \left[ I_Y + C(\lambda - A_W)^{-1} \tilde{B} \right] \\ &= I_Y + C(\lambda - A_W)^{-1} \tilde{B} - C(I - \Pi_W)(\lambda - A_W^\times)^{-1} \tilde{B} \\ &\quad + C(I - \Pi_W)(\lambda - A_W^\times)^{-1} [(\lambda - A_W) - (\lambda - A_W^\times)] (\lambda - A_W)^{-1} \tilde{B} \\ &= I_Y + C \Pi_W (\lambda - A_W)^{-1} \tilde{B} = W_{\theta_{r2}}(\lambda). \end{aligned} \quad (7.45)$$

If  $\theta_r$  is a right PS-realization, then it is not obvious if  $\theta_{r1}$  and  $\theta_{r2}^\times$  are right PS-realizations.

Now suppose that

$$\theta = (A, B, C; V, W, \tau; Y)$$

is an extended PS-realization such that  $W_\theta(\lambda)^{-1}$  is bounded on  $|\operatorname{Im} \lambda| \leq \varepsilon$ . Consider the corresponding left PS-realization  $\theta_l$  in (7.42) and the corresponding right PS-realization  $\theta_r$  in (7.44). Assuming that  $V = V_- \dot{+} V_+^\times$  and  $W = W_- \dot{+} W_+^\times$ ,  $\theta_{l1}$  and  $\theta_{l2}^\times$  are left PS-realizations and  $\theta_{r2}$  and  $\theta_{r1}^\times$  are right PS-realizations satisfying (7.43) and (7.45). Given that

$$(\lambda - A)^{-1}\tau = \tau(\lambda - A_W)^{-1}, \quad (\lambda - A^\times)^{-1}\tau = \tau(\lambda - A_W^\times)^{-1},$$

we clearly have

$$\tau[W_\pm] \subset V_\pm, \quad \tau[W_\pm^\times] \subset V_\pm^\times,$$

which implies

$$\Pi\tau = \tau\Pi_W.$$

Let  $\tau_1 : W_- \rightarrow V_-$  be the restriction of  $\tau$  to  $V_-$  and  $\tau_2 : W_+^\times \rightarrow V_+^\times$  the restriction of  $\tau$  to  $W_+^\times$ . Then  $\tau_1$  and  $\tau_2$  are continuous and dense imbeddings satisfying

$$(\lambda - A_j)^{-1}\tau_j = \tau_j(\lambda - A_{j,W})^{-1}, \quad (\lambda - A_j^\times)^{-1}\tau_j = \tau_j(\lambda - A_{j,W}^\times)^{-1}, \quad (7.46)$$

where  $j = 1, 2$ . As a result, for  $j = 1, 2$ ,

$$\begin{aligned} \tilde{C}_j(\lambda - A_j)^{-1}B_j &= \tilde{C}_j\tau_j(\lambda - A_{j,W})^{-1}\tilde{B}_j \\ &= \tilde{C}_j(\lambda - A_j)^{-1}\tau_j\tilde{B}_j = C_j(\lambda - A_{j,W})^{-1}\tilde{B}_j, \end{aligned} \quad (7.47a)$$

$$\begin{aligned} \tilde{C}_j(\lambda - A_j^\times)^{-1}B_j &= \tilde{C}_j\tau_j(\lambda - A_{j,W}^\times)^{-1}\tilde{B}_j \\ &= \tilde{C}_j(\lambda - A_j^\times)^{-1}\tau_j\tilde{B}_j = C_j(\lambda - A_{j,W}^\times)^{-1}\tilde{B}_j. \end{aligned} \quad (7.47b)$$

Consequently, for  $j = 1, 2$  and  $|\operatorname{Im} \lambda| \leq \varepsilon$  we have

$$W_{\theta_{lj}}(\lambda) = W_{\theta_{rj}}(\lambda), \quad W_{\theta_{lj}^\times}(\lambda) = W_{\theta_{rj}^\times}(\lambda).$$

Now (7.43) implies that

$$\begin{aligned} W_{\theta_{l1}}(\lambda)^{-1} &= W_{\theta_{r2}}(\lambda)W_\theta(\lambda)^{-1}, \\ W_{\theta_{r2}}(\lambda)^{-1} &= W_\theta(\lambda)^{-1}W_{\theta_{l1}}(\lambda)^{-1}, \end{aligned}$$

making either expression bounded on a strip  $|\operatorname{Im} \lambda| \leq \varepsilon$ . Then, according to Theorem 7.11,  $\theta_{l1}^\times$  and  $\theta_{l2}$  are left PS-realizations and  $\theta_{r2}^\times$  and  $\theta_{r1}$  are right PS-realizations. Using the intertwining relations (7.46) and (7.47) we see that  $\theta_1$  and  $\theta_2$  are extended PS-realizations satisfying (7.37).

**(3)  $\implies$  (2)** Let  $\theta = (A, B, C; V, W, \tau; Y)$  be an extended PS-realization and let (7.40) be uniquely solvable. Then

$$\Theta_l = (A, I_V, B\tilde{C}; V; V)$$

is a left PS-realization with transfer function  $(\lambda - A^\times)(\lambda - A)^{-1}$  and

$$\Theta_r = (A_W, \tilde{B}C, I_W; W; W)$$

is a right PS-realization with transfer function  $(\lambda - A_W)^{-1}(\lambda - A_W^\times)$ . Letting  $T_l^+ : L^2(\mathbb{R}^+; V) \rightarrow L^2(\mathbb{R}^+; Y)$  and  $T_r^+ : L^2(\mathbb{R}^+; Y) \rightarrow L^2(\mathbb{R}^+; W)$  stand for the bounded linear operators defined by

$$\begin{aligned} T_l^+ \phi &= -i \int_0^\infty \tilde{C}E(t-s; -iA)\phi(s) ds, \\ T_r^+ \phi &= -i \int_0^\infty E(t-s; -iA_W)\tilde{B}\phi(s) ds, \end{aligned}$$

where  $t \in \mathbb{R}^+$ , we see that  $T_l^+B$  and  $CT_r^+$  are the parts of the input-output operators  $T_{\theta_l}$  and  $T_{\theta_r}$  in  $L^2(\mathbb{R}^+; Y)$  and  $BT_l^+$  and  $T_r^+C$  are the parts of the input-output operators  $T_{\Theta_l}$  and  $T_{\Theta_r}$  in  $L^2(\mathbb{R}^+; V)$  and  $L^2(\mathbb{R}^+; W)$ , respectively. Thus the convolution equations

$$\Phi(t) + \int_0^\infty B\tilde{C}E(t-s; -iA)\Phi(s) ds = g(t), \quad t \in \mathbb{R}^+, \quad (7.48a)$$

$$\Psi(t) + \int_0^\infty E(t-s; -iA_W)\tilde{B}C\Psi(s) ds = h(t), \quad t \in \mathbb{R}^+, \quad (7.48b)$$

are uniquely solvable in  $L^2(\mathbb{R}^+; V)$  and  $L^2(\mathbb{R}^+; W)$ , respectively.

For  $x \in W$  let  $G(\cdot, x)$  be the unique solution of the convolution equation (7.48b) with right-hand side  $g(t) = E(t; -iA_W)x$ . Then for  $u \geq 0$  and  $t > 0$  we have

$$\begin{aligned} G(t+u, x) &- i \int_0^\infty E(t-s; -iA_W)\tilde{B}CG(s+u, x) ds \\ &= G(t+u, x) - i \int_u^\infty E(t+u-s; -iA_W)\tilde{B}CG(s, x) ds \\ &= E(t+u; -iA_W)x + i \int_0^u E(t+u-s; -iA_W)\tilde{B}CG(s, x) ds \\ &= E(t; -iA_W) \left[ E(u; -iA_W)x + i \int_0^u E(u-s; -iA_W)\tilde{B}CG(s, x) ds \right] \\ &= E(t; -iA_W) \left[ E(u; -iA_W)x + i \int_0^\infty E(u-s; -iA_W)\tilde{B}CG(s, x) ds \right] \\ &= E(t; -iA_W)G(u, x). \end{aligned}$$



Thus we have found the product rule

$$G(t+u, x) = G(t, G(u, x)), \quad x \in W, \quad t, u \in \mathbb{R}^+.$$

For  $t \in \mathbb{R}^+$  let us introduce the bounded linear operator  $G(t)$  defined by  $G(t)x = G(t, x)$  for  $x \in W$ . Then  $G(0)$  is a bounded projection on  $W$  with kernel  $W_-$ .

Let us now substitute  $E(\cdot; -iA_W^\times)G(0)x$  with  $x \in W$  in the left-hand side of (7.48b), yielding  $e \in L^2(\mathbb{R}^+; W)$ . Extending the convolution equation to the full line in the usual way and taking Fourier transforms we obtain

$$\hat{e}_+(\lambda) = (\lambda - A_W)^{-1}E(0^+; -iA_W)x.$$

Therefore,

$$W_- \dot{+} \text{Im } G(0) = W, \quad \text{Im } G(0) \subset W_+^\times.$$

Similarly, substituting  $E(\cdot; -iA_W^\times)x$  with  $x \in W$  in the left-hand side of (7.48b) we get  $E(\cdot; -iA_W)x$  on the right-hand side. Thus  $W_- \dot{+} W_+^\times = W$ . From the unique solvability of (7.48b) it now follows that

$$W_- \dot{+} W_+^\times = W,$$

so that  $G(0)$  is the projection of  $W$  onto  $W_+^\times$  along  $W_-$ .

Taking the limit in (7.48b) as  $t \rightarrow 0^+$  we get, for  $x \in W$ ,

$$\begin{aligned} \tau G(0)x &= \tau E(0^+; -iA_W)x + i \int_0^\infty E(-s; -iA_W) \tilde{B} C E(s; -iA_W^\times) G(0)x ds \\ &= E(0^+; -iA) \tau x - i \Gamma_{\theta_r} U \Lambda_{\theta_l^\times} \tau G(0)x, \end{aligned}$$

where  $U : L^2(\mathbb{R}; Y) \rightarrow L^2(\mathbb{R}^-; Y)$  is given by  $(U\phi)(t) = \phi(-t)$ . As a result,

$$\|\tau G(0)x\|_V \leq \text{const.} \|\tau x\|_V, \quad x \in W,$$

implying the boundedness of  $\tau G(0)\tau^{-1}$  on  $V$ . Thus  $\tau G(0)\tau^{-1}$  extends to the projection of  $V$  onto  $V_+^\times$  along  $V_-$  and hence  $V_+^\times \dot{+} V_- = V$ .  $\square$

We now prove the analog of Corollary 7.5 for extended PS-realizations.

**Corollary 7.14.** *Let  $\theta = (A, B, C; V, W, \tau; Y)$  be an extended PS-realization satisfying*

$$\sup_{|\text{Im } \lambda| \leq \mu} \|W_\theta(\lambda) - I_Y\|_{\mathcal{L}(Y)} < 1$$

*for some  $\mu \in (0, \omega_\theta)$ . Then  $W_\theta(\cdot)$  has a left and a right quasi-canonical factorization.*

The corollary is immediate from (7.28) which implies that

$$\|\tilde{T}_\theta\|_{\mathcal{L}(L^2(\mathbb{R}^+; Y))} \leq \|T_\theta\|_{\mathcal{L}(L^2(\mathbb{R}^+; Y))} = \sup_{\lambda \in \mathbb{R}} \|W_\theta(\lambda) - I_Y\|_{\mathcal{L}(Y)} < 1.$$

This in turn implies the unique solvability of (7.41) on  $L^2(\mathbb{R}^+; Y)$ .

It is not obvious how to derive general quasi-canonical factorization theorems for left PS-realizations and right PS-realizations, especially how to prove the transition **(2)**  $\implies$  **(1)**. As we have seen above, it is straightforward to prove that  $\theta_{l1}$  and  $\theta_{l2}^\times$  are left PS-realizations satisfying (7.43) if  $\theta_l$  is a left PS-realization, and that  $\theta_{r2}$  and  $\theta_{r1}^\times$  are right PS-realizations satisfying (7.45) if  $\theta_r$  is a right PS-realization. To complete the proof, we need to apply Theorem 7.11 to  $\theta_{l1}$ ,  $\theta_{l2}^\times$ ,  $\theta_{r2}$ , and  $\theta_{r1}^\times$ , but this requires the boundedness of the inverses of their respective symbols on a horizontal strip. In the case of extended PS-realizations (Theorem 7.13) this easily follows by equating the transfer functions of  $\tau$ -compatible left and right PS-realizations. Below in Theorems 7.15 and 7.16 we assume the compactness of the bounded operator  $B$  or  $C$  in the realization to arrive at a similar result. In this way we generalize the factorization result given in [16] for so-called BGK realizations with a finite-dimensional input-output space  $Y$ .

**Theorem 7.15.** *Let  $\theta_l = (A, B, \tilde{C}; V; Y)$  be a left PS-realization such that  $B$  is a compact operator. Then the following statements are equivalent:*

1.  $W_{\theta_l}(\cdot)$  has a left canonical factorization.
2. We have the decomposition  $V = V_- \dot{+} V_+^\times$ .
3. For every  $g \in L^2(\mathbb{R}^+; Y)$  the convolution equation

$$\phi(t) + \int_0^\infty k_\theta(t-s, \phi(s)) ds = g(t), \quad t \in \mathbb{R}^+, \quad (7.49)$$

is uniquely solvable in  $L^2(\mathbb{R}^+; Y)$ .

Moreover, we have the equivalent statements:

- 1'.  $W_{\theta_l}(\cdot)$  has a right canonical factorization.
- 2'. We have the decomposition  $V = V_+ \dot{+} V_-^\times$ .
- 3'. For every  $g \in L^2(\mathbb{R}^-; Y)$  the convolution equation

$$\phi(t) + \int_{-\infty}^0 k_{\theta_l}(t-s, \phi(s)) ds = g(t), \quad t \in \mathbb{R}^-, \quad (7.50)$$

is uniquely solvable in  $L^2(\mathbb{R}^-; Y)$ .

In (7.49)–(7.50) the integrals are to be understood as Pettis integrals.

*Proof.* If  $\theta_l = (A, B, \tilde{C}; V; Y)$  is a left PS-realization, then for some  $\varepsilon > 0$  we have, as a consequence of the Riemann-Lebesgue Lemma,

$$\lim_{\substack{|\lambda| \rightarrow +\infty \\ |\operatorname{Im} \lambda| \leq \varepsilon}} \|\tilde{C}(\lambda - A)^{-1}x\|_Y = 0, \quad x \in V,$$

because for each  $x \in V$ ,

$$\tilde{C}(\lambda - A)^{-1}x = -i \int_{-\infty}^\infty e^{i\lambda t} (\Lambda_{\theta_l} x)(t) dt, \quad e^{\varepsilon|\cdot|} \Lambda_{\theta_l} x \in L^1(\mathbb{R}; Y).$$

Then the compactness of  $B$  implies that

$$\|W_{\theta_i}(\lambda) - I_Y\|_{\mathcal{L}(Y)} = \|\tilde{C}(\lambda - A)^{-1}B\|_{\mathcal{L}(Y)}$$

vanishes as  $|\lambda| \rightarrow +\infty$  within the strip  $|\operatorname{Im} \lambda| \leq \varepsilon$ .

**(2)  $\implies$  (1)** Now assume that  $V = V_- \dot{+} V_+^\times$ . Then the compactness of  $B$  implies that  $W_{\theta_{i1}}(\lambda)$  and  $W_{\theta_{i2}^\times}(\lambda)$  tend to the identity in the operator norm as  $|\lambda| \rightarrow +\infty$  within this strip and hence that the inverses of these operator functions are bounded on a (possibly reduced) strip. We may then apply Theorem 7.11 to prove that  $\theta_{i1}^\times$  and  $\theta_{i2}$  are left PS-realizations as well. The factorization

$$W_{\theta_i}(\lambda) = W_{\theta_{i1}}(\lambda)W_{\theta_{i2}}(\lambda)$$

then is a left canonical factorization.

**(3)  $\implies$  (2)** Let (7.49) be uniquely solvable. Then (7.48a) is uniquely solvable. Further, it is easily shown that, for each  $x \in V$ ,

$$\Phi(t) = B\tilde{C}E(t; -iA^\times)x \quad (7.51)$$

is the solution corresponding to the right-hand side  $g(t) = B\tilde{C}E(t; -iA)x$ . Suppose  $x \in V_- \cap V_+^\times$ . Then (7.51) would be a solution of (7.48a) with zero right-hand side and hence the zero solution. Taking their Fourier transforms, we get

$$\underbrace{(\lambda - A^\times)^{-1}x}_{\text{analytic for } \operatorname{Im} \lambda > -\varepsilon} = \underbrace{(\lambda - A)^{-1}x}_{\text{analytic for } \operatorname{Im} \lambda < \varepsilon}, \quad \lambda \in \mathbb{R},$$

which implies  $x = 0$ , by Liouville's theorem. Thus  $V_- \cap V_+^\times = \{0\}$ .

Now let  $\Phi(\cdot, x)$  be the unique solution of (7.48a) with right-hand side  $g(t) = B\tilde{C}E(t; -iA)x$ . Then for a.e.  $t \in \mathbb{R}^+$  and each  $u \in \mathbb{R}^+$  we compute

$$\begin{aligned} \Phi(t+u, x) - i \int_0^\infty B\tilde{C}E(t-s; -iA)\Phi(s+u, x) ds \\ &= \Phi(t+u, x) - i \int_u^\infty B\tilde{C}E(t+u-s; -iA)\Phi(s, x) ds \\ &= B\tilde{C}E(t+u; -iA)x + i \int_0^u B\tilde{C}E(t+u-s; -iA)\Phi(s, x) ds \\ &= B\tilde{C}E(t; -iA) \left[ E(u; -iA)x + i \int_0^u E(u-s; -iA)\Phi(s, x) ds \right] \\ &= B\tilde{C}E(t; -iA) \left[ E(u; -iA)x + i \int_0^\infty E(u-s; -iA)\Phi(s, x) ds \right] \\ &= B\tilde{C}E(t; -iA)F(u, x), \end{aligned}$$

where  $F(\cdot, x) \in BC(\mathbb{R}^+; V)$  for every  $u \in \mathbb{R}^+$ , thus justifying the third of the above transitions. Letting  $u \rightarrow 0^+$  we get  $\Phi(t, x) = \tilde{B}\tilde{C}E(t; -iA)x$  for a.e.  $t \in \mathbb{R}^+$ . The existence of a solution of (7.48a) for  $g(t) = E(t; -iA)x$  then implies that

$$[V_+^\times + V_-] \cap V_+ = V_+.$$

Using that

$$\frac{V}{V_+^\times + V_-} = \frac{[V_+^\times + V_-] + V_+}{V_+^\times + V_-} \simeq \frac{V_+}{[V_+^\times + V_-] \cap V_+} = \{0\}$$

as complex vector space isomorphisms, we get  $V_+^\times + V_- = V$ , as claimed.  $\square$

Analogously we have

**Theorem 7.16.** *Let  $\theta_r = (A_W, \tilde{B}, C; W; Y)$  be a right PS-realization such that  $C$  is a compact operator. Then the following statements are equivalent:*

1.  $W_{\theta_r}(\cdot)$  has a left canonical factorization.
2. We have the decomposition  $W = W_- \dot{+} W_+^\times$ .
3. For every  $g \in L^2(\mathbb{R}^+; Y)$  the convolution equation

$$\phi(t) + \int_0^\infty k_{\theta_r}(t-s, \phi(s)) ds = g(t), \quad t \in \mathbb{R}^+, \quad (7.52)$$

is uniquely solvable in  $L^2(\mathbb{R}^+; Y)$ .

Moreover, we have the equivalent statements:

- 1'.  $W_{\theta_r}(\cdot)$  has a right canonical factorization.
- 2'. We have the decomposition  $W = W_+ \dot{+} W_-^\times$ .
- 3'. For every  $g \in L^2(\mathbb{R}^-; Y)$  the convolution equation

$$\phi(t) + \int_{-\infty}^0 k_{\theta_r}(t-s, \phi(s)) ds = g(t), \quad t \in \mathbb{R}^-, \quad (7.53)$$

is uniquely solvable in  $L^2(\mathbb{R}^-; Y)$ .

In (7.52)–(7.53) the integrals are to be understood as Pettis integrals.

*Proof.* If  $\theta_r = (A_W, \tilde{B}, C; W; Y)$  is a right PS-realization, then (7.15) and (7.16) imply that

$$\lim_{\substack{|\lambda| \rightarrow +\infty \\ |\operatorname{Im} \lambda| \leq \varepsilon}} \left\| [(\lambda - A_W)^{-1} \tilde{B}]^* w \right\|_Y = 0, \quad w \in W.$$

The compactness of  $C$  then implies that

$$\|W_{\theta_r}(\lambda) - I_Y\|_{\mathcal{L}(Y)} = \|C(\lambda - A_W)^{-1} \tilde{B}\|_{\mathcal{L}(Y)}$$

vanishes as  $|\lambda| \rightarrow +\infty$  within the strip  $|\operatorname{Im} \lambda| \leq \varepsilon$ .

(2)  $\implies$  (1) Now assume that  $W = W_- + W_+^\times$ . Then the compactness of  $C$  implies that  $W_{\theta_{r2}}(\lambda)$  and  $W_{\theta_{r1}^\times}(\lambda)$  tend to the identity in the operator norm as  $|\lambda| \rightarrow +\infty$  within a strip and hence that the inverses of these operator functions are bounded on a strip. We may then apply Theorem 7.11 to prove that  $\theta_{r2}^\times$  and  $\theta_{r1}$  are right PS-realizations as well. The factorization

$$W_{\theta_r}(\lambda) = W_{\theta_{r1}}(\lambda)W_{\theta_{r2}}(\lambda)$$

then is a left canonical factorization.

(3)  $\implies$  (2) As in the proof of Theorem 7.13, we use the unique solvability of (7.48b). We may then repeat its proof almost verbatim and conclude that  $W = W_- + W_+^\times$ . The proof of (1)  $\implies$  (3) is again trivial.  $\square$

### 7.2.5 Solving the realization problem

In this subsection we give a complete description of the class of operator functions which can be represented as transfer functions of extended PS-realizations. We actually solve the so-called *realization problem*, i.e., we explicitly construct the extended PS-realization whose transfer function coincides with the given operator function. In the final paragraph of this subsection we also solve the realization problem for right PS-realizations. There is no known solution of the realization problem for left PS-realizations.

The realization problem for noncausal linear systems with all three operators  $A$ ,  $B$ , and  $C$  bounded, which leads to a  $C^\infty$  weighting pattern  $k$  satisfying  $e^{\varepsilon|\cdot|}k(\cdot) \in L^1(\mathbb{R}; \mathcal{L}(Y))$ , has been solved by Bart and Kroon [20]. For left PS-realizations on finite-dimensional spaces the realization problem was solved by Bart, Gohberg, and Kaashoek [17]. We now present the solution of the realization problem for extended PS-realizations as in [97].

Given  $\mu \geq 0$  and a complex Hilbert space  $Y$ , we let  $L_{\pm\mu}^2(\mathbb{R}^+; Y)$  stand for the complex Hilbert spaces of all strongly measurable functions  $\phi: \mathbb{R}^+ \rightarrow Y$  which are bounded with respect to the norm

$$\|\phi\| = \left[ \int_0^\infty e^{\mp 2\mu t} \|\phi(t)\|_Y^2 dt \right]^{1/2}.$$

Similarly, we let  $L_{\pm\mu}^2(\mathbb{R}^-; Y)$  be the complex Hilbert spaces of all strongly measurable functions  $\phi: \mathbb{R}^- \rightarrow Y$  which are bounded with respect to the norm

$$\|\phi\| = \left[ \int_{-\infty}^0 e^{\mp 2\mu t} \|\phi(t)\|_Y^2 dt \right]^{1/2}.$$

Finally, we define  $L_{\pm\mu}^2(\mathbb{R}; Y)$  to be the complex Hilbert spaces of all strongly measurable functions  $\phi: \mathbb{R} \rightarrow Y$  which are bounded with respect to the norm

$$\|\phi\| = \left[ \int_{-\infty}^\infty e^{\mp 2\mu|t|} \|\phi(t)\|_Y^2 dt \right]^{1/2}.$$

In the sense of continuous and, as far as the horizontal arrows are concerned, dense imbeddings we have

$$\begin{array}{ccccc}
 L^2_{-\mu}(\mathbb{R}^+; Y) & \longrightarrow & L^2(\mathbb{R}^+; Y) & \longrightarrow & L^2_{\mu}(\mathbb{R}^+; Y) \\
 \downarrow & & \downarrow & & \downarrow \\
 L^2_{-\mu}(\mathbb{R}; Y) & \longrightarrow & L^2(\mathbb{R}; Y) & \longrightarrow & L^2_{\mu}(\mathbb{R}; Y) \\
 \uparrow & & \uparrow & & \uparrow \\
 L^2_{-\mu}(\mathbb{R}^-; Y) & \longrightarrow & L^2(\mathbb{R}^-; Y) & \longrightarrow & L^2_{\mu}(\mathbb{R}^-; Y).
 \end{array}$$

We now state the main result.

**Theorem 7.17.** *Suppose there exist  $\mu > 0$ ,  $k : \mathbb{R} \times Y \rightarrow Y$ , and  $k_* : \mathbb{R} \times Y$  having the following properties:*

1.  $\{k(\cdot, u), k_*(\cdot, y)\} \subset L^2_{-\mu}(\mathbb{R}; Y)$  for all  $u, y \in Y$ .
2.  $\langle k(\cdot, u), y \rangle = \langle u, k_*(\cdot, y) \rangle$  for all  $u, y \in Y$ .
3. For  $|\operatorname{Im} \lambda| < \mu$  and  $u, y \in Y$  we have

$$W(\lambda)u = u + \int_{-\infty}^{\infty} e^{i\lambda t} k(t, u) dt, \quad W(\bar{\lambda})^* y = y + \int_{-\infty}^{\infty} e^{-i\lambda t} k_*(t, y) dt.$$

Then  $W$  is the transfer function of an extended PS-realization  $\theta$  satisfying  $\omega_{\theta} \geq \mu$ .

It should be noted that by slightly increasing  $\mu > 0$  we can make the operator function  $W$  in the statement of Theorem 7.17 satisfy the following additional condition:

4. For every  $z \in Y$  we have

$$\int_{-\infty}^{\infty} e^{\mu|t|} (\|k(t, z)\|_Y + \|k_*(t, z)\|_Y) dt < +\infty.$$

On the other hand, let  $\theta = (A, B, C; V, W, \tau; Y)$  be an extended PS-realization. Then also  $\theta^* = (A_W^*, C^*, B^*; W, V, \tau^*; Y)$  is an extended PS-realization and the above conditions 1–4 are satisfied for any  $\mu \in (0, \omega_{\theta})$ .

We now derive the following auxiliary result.

**Lemma 7.18.** *Let  $\mu > 0$ . Suppose  $Y$  is a complex Hilbert space and  $k : \mathbb{R}^+ \times Y \rightarrow Y$  and  $k_* : \mathbb{R}^+ \times Y \rightarrow Y$  satisfy the following two conditions:*

- a.  $\{e^{\mu \cdot} k(\cdot, u), e^{\mu \cdot} k_*(\cdot, y)\} \subset L^1(\mathbb{R}^+; Y) \cap L^2(\mathbb{R}^+; Y)$  for all  $u, y \in Y$ .
- b. For all  $u, y \in Y$  we have  $\langle k(\cdot, u), y \rangle = \langle u, k_*(\cdot, y) \rangle$ .

Then the linear operator  $H$  defined by

$$(H\phi)(t) = \int_0^\infty k(t+s, \phi(s)) ds, \quad t \in \mathbb{R}^+ \text{ a.e.}, \quad (7.54)$$

is bounded from  $L_\mu^2(\mathbb{R}^+; Y)$  into  $L_{-\mu}^2(\mathbb{R}^+; Y)$  and also from  $L^2(\mathbb{R}^+; Y)$  into itself.

The integral in (7.54) is to be understood as a Pettis integral in the following sense:

$$\langle (H\phi)(t), y \rangle = \int_0^\infty \langle k(t+s, \phi(s)), y \rangle ds = \int_0^\infty \langle \phi(s), k_*(t+s, y) \rangle ds,$$

where  $y \in Y$  and  $t \in \mathbb{R}^+$  a.e.

*Proof.* For any  $\nu \in [-\mu, \mu]$  the map  $J_\nu : Y \rightarrow L^1(\mathbb{R}^+; Y)$  defined by  $Ju = e^{\nu \cdot} k(\cdot, u)$  is bounded. Indeed, letting  $\{u_n\}_{n=1}^\infty$  be a sequence satisfying

$$\lim_{n \rightarrow \infty} \|u_n - u\|_U = 0, \quad \lim_{n \rightarrow \infty} \|e^{\nu \cdot} Ju_n - \phi\|_{L^1(\mathbb{R}^+; Y)} = 0,$$

where  $u \in U$  and  $\phi \in L^1(\mathbb{R}^+; Y)$ , we see that for each  $y \in Y$  the scalar functions

$$e^{\nu \cdot} \langle u_n, k_*(\cdot, y) \rangle = e^{\nu \cdot} \langle k(\cdot, u_n), y \rangle \in L^1(\mathbb{R}^+),$$

while

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty e^{\nu t} |(u_n, k_*(t, y)) - (u, k_*(t, y))| dt &= 0, \\ \lim_{n \rightarrow \infty} \int_0^\infty |e^{\nu t} (k(\cdot, u_n), y) - (\phi(t), y)| dt &= 0, \end{aligned}$$

and the left-hand sides are obviously equal. Therefore,

$$e^{\nu t} \langle u, k_*(t, y) \rangle = \langle \phi(t), y \rangle, \quad y \in Y, \quad t \in \mathbb{R}^+ \text{ a.e.},$$

whence  $\phi = e^{\nu \cdot} k(\cdot, u)$ . But this shows  $J_\nu$  to be a closed operator and hence, by the Closed Graph Theorem,  $J_\nu$  is bounded:

$$\|e^{\nu \cdot} k(\cdot, u)\|_{L^1(\mathbb{R}^+; Y)} \leq \gamma(\nu) \|u\|_Y, \quad u \in Y, \quad \nu \in [-\mu, \mu]. \quad (7.55)$$

This boundedness property implies in particular that the Pettis integral in (7.54) is well defined.

Put

$$\phi(t) = \begin{cases} \phi_j, & t \in E_j, \quad j = 1, \dots, r, \\ 0, & \text{otherwise,} \end{cases}$$

where  $E_1, \dots, E_r$  are mutually disjoint subsets of  $\mathbb{R}^+$  of finite measure. Then by the hypotheses on  $k$  we have

$$H\phi = \sum_{j=1}^r \int_{E_j} k(\cdot + s, \phi_j) ds \in L^2(\mathbb{R}^+; Y).$$

Moreover,  $H$  can be viewed as the composition product of the following three operators: 1) the sign flip  $\phi(t) \mapsto \phi(-t)$ , 2) the convolution operator mapping  $\phi$  to  $\int_{-\infty}^{\infty} k(\cdot - s, \phi(s)) ds$ , and 3) the restriction to the positive half-line, as illustrated by the diagram

$$L^2(\mathbb{R}^+; Y) \xrightarrow{\text{sign flip}} L^2(\mathbb{R}; Y) \xrightarrow{\text{convolution}} L^2(\mathbb{R}; Y) \xrightarrow{\text{restriction}} L^2(\mathbb{R}^+; Y).$$

In view of (7.55) with  $\nu = 0$ , we can apply Lemma 2.11 to prove that the convolution operator in the diagram is a bounded linear operator on  $L^2(\mathbb{R}; Y)$ . Consequently,  $H$  is a bounded linear operator on  $L^2(\mathbb{R}^+; Y)$ .

Replacing  $k$  by  $e^{\mu \cdot} k$  it is easily seen using the same arguments that  $H$  is bounded as a linear operator from  $L_{\mu}^2(\mathbb{R}^+; Y)$  into  $L_{-\mu}^2(\mathbb{R}^+; Y)$ .  $\square$

Let us now prove Theorem 7.17 as follows. Using  $k(t, \cdot)$  and  $k_*(t, \cdot)$  for  $t \in \mathbb{R}^+$  a.e., we now construct the “forward” part of the extended PS-realization. In other words, we construct an extended PS-realization with weighting pattern  $k$  as if  $k(t, \cdot) = 0$  and  $k_*(t, \cdot) = 0$  for  $t \in \mathbb{R}^-$  a.e. Here we follow [97]. After that, we repeat the construction with  $k(t, \cdot)$  and  $k_*(t, \cdot)$  replaced by  $k(-t, \cdot)$  and  $k(-t, \cdot)$  for  $t \in \mathbb{R}^+$  a.e. Finally, we rearrange the two extended PS-realizations obtained to construct a single extended PS-realization with weighting pattern  $k$ .

*Proof of Theorem 7.17.* Indeed, let us first disregard  $k(t, \cdot)$  and  $k_*(t, \cdot)$  for  $t \in \mathbb{R}^-$ . For  $t > 0$  we define

$$\begin{aligned} [E(t)f](s) &= f(t+s), & t, s > 0, \\ [E^{\#}(t)f](s) &= \begin{cases} f(s-t), & s > t > 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Then  $E(\cdot)$  induces strongly continuous semigroups on the Hilbert spaces

$$L_{-\mu}^2(\mathbb{R}^+; Y) \quad \text{and} \quad L_{\mu}^2(\mathbb{R}^+; Y).$$

We denote these semigroups by  $E_-(\cdot)$  and  $E_+(\cdot)$ , respectively. Similarly,  $E_-^{\#}(\cdot)$  and  $E_+^{\#}(\cdot)$  are the strongly continuous semigroups induced by  $E^{\#}(\cdot)$  on  $L_{-\mu}^2(\mathbb{R}^+; Y)$  and  $L_{\mu}^2(\mathbb{R}^+; Y)$ , respectively. The semigroups  $E_-(\cdot)$  and  $E_+^{\#}(\cdot)$  are both exponentially decaying. In fact,  $E_-(t)$  and  $E_+^{\#}(t)$  both have norm  $e^{-\mu t}$ . Further,

$$E_-(t)^* = e^{-2\mu t} E_-^{\#}(t), \quad E_+(t)^* = e^{2\mu t} E_+^{\#}(t).$$



Now let  $H$  be as in Lemma 7.18. Then

$$E_-(t)H = HE_+^\#(t), \quad E_+(t)H^* = H^*E_-^\#(t),$$

where  $t \geq 0$ . Thus  $H[L_\mu^2(\mathbb{R}^+; Y)]$  is invariant under  $E_-(t)$ . We define  $V$  to be the closure of  $H[L_\mu^2(\mathbb{R}^+; Y)]$  in  $L_{-\mu}^2(\mathbb{R}^+; Y)$ . Then  $E_-(t)|_V$  is an exponentially decaying strongly continuous semigroup on  $V$  whose generator will be denoted by  $-iA$ .

Next, let  $Q$  be the orthogonal projection of  $L_\mu^2(\mathbb{R}^+; Y)$  along  $\text{Ker } H$ . Since  $\text{Ker } H$  is invariant under  $E_+^\#(t)$ , we have  $QE_+^\#(t) = QE_+^\#(t)Q$  for each  $t \in \mathbb{R}^+$ . If  $H\psi = \phi$  for some  $\psi \in L_\mu^2(\mathbb{R}^+; Y)$ , then the vector  $Q\psi$  is completely determined by  $\phi$ . We define  $W$  to be the complex Hilbert space obtained by endowing  $\text{Im } H \subset L_{-\mu}^2(\mathbb{R}^+; Y)$  with the norm

$$\|\phi\|_W = \left[ \|\phi\|_{L_{-\mu}^2(\mathbb{R}^+; Y)}^2 + \|Q\psi\|_{L_\mu^2(\mathbb{R}^+; Y)}^2 \right]^{1/2},$$

where  $\psi$  is some vector such that  $H\psi = \phi$ .<sup>2</sup> If  $H\psi = \phi$ , then  $E_-(t)\phi = HE_+^\#(t)\psi$ . Since  $QE_+^\#(t)\psi = QE_+^\#(t)Q\psi$ , we see that

$$\begin{aligned} \|E_-(t)\phi\|_W^2 &= \|E_-(t)\phi\|_{L_{-\mu}^2(\mathbb{R}^+; Y)}^2 + \|QE_+^\#(t)Q\psi\|_{L_\mu^2(\mathbb{R}^+; Y)}^2 \\ &\leq e^{-\mu t} \|\phi\|_{L_{-\mu}^2(\mathbb{R}^+; Y)}^2 + e^{-\mu t} \|Q\psi\|_{L_\mu^2(\mathbb{R}^+; Y)}^2 = e^{-\mu t} \|\phi\|_W^2. \end{aligned}$$

Thus  $E_-(t)$  induces an exponentially decaying strongly continuous semigroup on  $W$ . Let  $\tau : W \rightarrow V$  be the natural imbedding of  $W$  into  $V$ . Then  $\tau[W]$  is dense in  $V$ . Writing  $A_W$  for the part of  $A$  in  $W$ , we see that  $E(t; -iA_W)$  is the restriction of  $E_-(t)$  to  $W$ .

Now define

$$(Bu)(t) = k(t, u), \quad u \in Y, \quad t \in \mathbb{R}^+ \text{ a.e.},$$

$$C\phi = \int_0^\infty k(t, (Q\psi)(t)) dt, \quad \phi = H\psi,$$

where the integral is to be defined as a Pettis integral. Then

$$\theta = (A, B, C; V, W, \tau; Y)$$

is an extended PS-realization whose weighting pattern is the restriction of  $k$  to  $\mathbb{R}^+ \times Y$ . Indeed, define

$$(B_n u)(t) = n \int_0^{1/n} k(t+s, u) ds, \quad t \in \mathbb{R}^+ \text{ a.e.}$$

---

<sup>2</sup>If  $\text{Ker } H = \{0\}$ , then  $W$  coincides with  $\text{Im } H$  endowed with the graph norm corresponding to  $H^{-1}$ .

Then  $B_n u \in \text{Im } H$ , while

$$\|B_n u - Bu\|_{L^2_{-\mu}(\mathbb{R}^+; Y)}^2 = \int_0^\infty e^{2\mu t} \left\| n \int_0^{1/n} \{k(t+s, u) - k(t, u)\} ds \right\|^2 dt.$$

Now

$$\begin{aligned} & \left\| n \int_0^{1/n} \{k(t+s, u) - k(t, u)\} ds \right\|^2 \leq \left\{ n \int_0^{1/n} \|k(t+s, u) - k(t, u)\| ds \right\}^2 \\ & \leq n^2 \left( \int_0^{1/n} \|k(t+s, u) - k(t, u)\|^2 ds \right) \int_0^{1/n} 1 ds \\ & = n \int_0^{1/n} \|k(t+s, u) - k(t, u)\|^2 ds. \end{aligned}$$

Then

$$\begin{aligned} & \int_0^\infty e^{2\mu t} \left\| n \int_0^{1/n} \{k(t+s, u) - k(t, u)\} ds \right\|^2 dt \\ & \leq \int_0^\infty \left\{ n \int_0^{1/n} e^{2\mu t} \|k(t+s, u) - k(t, u)\|^2 ds \right\} dt \\ & = n \int_0^{1/n} \left( \int_0^\infty e^{2\mu t} \|k(t+s, u) - k(t, u)\|^2 dt \right) ds. \end{aligned}$$

As  $\int_0^\infty e^{2\mu t} \|k(t+s, u) - k(t, u)\|^2 dt$  is continuous in  $s$  we see that  $B_n u$  tends to  $Bu$  in the norm of  $L^2_{-\mu}(\mathbb{R}^+; Y)$ . Thus  $B \in \mathcal{L}(Y, V)$ .

Take  $\phi \in L^2(\mathbb{R}^+; Y)$ . As  $E(\cdot; -iA)$  is an exponentially decaying semigroup, it follows that  $E(t; -iA)B\phi(t) \in L^1(\mathbb{R}^+; V)$ . Moreover,  $B\phi(t) \in L^2_{-\mu}(\mathbb{R}^+; Y)$ . So  $E(t; -iA)B\phi(t) = E_-(t)k(\cdot, \phi(t))$ , implying

$$\begin{aligned} \int_0^\infty E(t; -iA)B\phi(t) dt &= \int_0^\infty E_-(t)k(\cdot, \phi(t)) dt \\ &= \int_0^\infty k(\cdot + t, \phi(t)) dt = H\phi \in W. \end{aligned}$$

Therefore, for  $\phi \in L^2(\mathbb{R}^+; Y)$  we have

$$\begin{aligned} & \left\| \int_0^\infty E(t; -iA)B\phi(t) dt \right\|_W^2 = \|H\phi\|_W^2 = \|H\phi\|_{L^2_{-\mu}(\mathbb{R}^+; Y)}^2 + \|Q\phi\|_{L^2_\mu(\mathbb{R}^+; Y)}^2 \\ & \leq (\gamma^2 + 1)\|\phi\|_{L^2_\mu(\mathbb{R}^+; Y)}^2 \leq \tilde{\gamma}^2\|\phi\|_{L^2(\mathbb{R}^+; Y)}^2 \end{aligned}$$

for some constants  $\gamma$  and  $\tilde{\gamma}$ . To get the penultimate transition we used that  $H$  is bounded (by Lemma 7.18) and that  $Q$  is also bounded.

On the other hand, if  $\phi = H\psi$  for some  $\psi \in L^2_\mu(\mathbb{R}^+; Y)$ , we have

$$\begin{aligned} \|C\phi\|_Y &= \sup_{\|y\|_Y=1} |\langle C\phi, y \rangle| \leq \sup_{\|y\|_Y=1} \int_0^\infty |\langle (Q\psi)(t), k_*(t, y) \rangle| dt \\ &\leq \sup_{\|y\|_Y=1} \left[ \int_0^\infty e^{2\mu t} \|k_*(t, y)\|^2 dt \right]^{1/2} \left[ \int_0^\infty e^{-2\mu t} \|(Q\psi)(t)\|^2 dt \right]^{1/2} \\ &\leq \text{const.} \|Q\psi\|_{L^2_\mu(\mathbb{R}^+; Y)} \leq \text{const.} \|\phi\|_W, \quad \phi \in \text{Im } H, \end{aligned}$$

so that  $C \in \mathcal{L}(W, Y)$ .

We also have

$$CH\psi = \int_0^\infty k(s, \psi(s)) ds.$$

Indeed, for  $y \in Y$  we have

$$\begin{aligned} \langle (H\psi)(t), y \rangle &= \int_0^\infty \langle k(t+s, \psi(s)), y \rangle ds \\ &= \int_0^\infty \langle \psi(s), k_*(t+s, y) \rangle ds \\ &= \int_0^\infty \langle \psi(s), (E_-(t)k_*(\cdot, y))(s) \rangle ds = \langle \psi, E_-(t)k_*(\cdot, y) \rangle, \end{aligned}$$

which, as  $t \rightarrow 0^+$ , tends to  $\langle \psi, k_*(\cdot, y) \rangle = \langle \int_0^\infty k(s, \psi(s)) ds, y \rangle$ . Thus

$$\begin{aligned} \langle CH\psi, y \rangle &= \left\langle \int_0^\infty k(s, (Q\psi)(s)) ds, y \right\rangle \\ &= \lim_{t \rightarrow 0^+} \langle (HQ\psi)(t), y \rangle = \lim_{t \rightarrow 0^+} \langle (H\psi)(t), y \rangle, \end{aligned}$$

because  $H(I - Q) = 0$ , and so  $\langle CH\psi, y \rangle = \langle \int_0^\infty k(s, \psi(s)) ds, y \rangle$ . Hence,  $CH\psi = \int_0^\infty k(s, \psi(s)) ds$ . As a result, we have for  $\phi \in W$ ,

$$\begin{aligned} CE(t; -iA_W)\phi &= CE_-(t)H\psi = CHE^\#(t)\psi \\ &= \int_0^\infty k(s, (E^\#(t)\psi)(s)) ds \\ &= \int_t^\infty k(s, \psi(s-t)) ds = (H\psi)(t) = \phi(t), \end{aligned}$$

whence  $CE(\cdot, -iA_W)\phi \in L^2(\mathbb{R}^+; Y)$  and

$$\|CE(\cdot, -iA_W)\phi\|_{L^2(\mathbb{R}^+; Y)} = \|\phi\|_{L^2(\mathbb{R}^+; Y)} \leq \|\phi\|_{L^2_{-\mu}(\mathbb{R}^+; Y)},$$

where  $\phi \in L^2_{-\mu}(\mathbb{R}^+; Y)$ . Consequently,  $\theta$  is an extended PS-realization.

Finally, since the weighting pattern  $k_\theta$  of  $\theta$  is given by  $k_\theta(\cdot, u) = -i\Lambda_\theta Bu$ , it is straightforward to see that  $k_\theta$  and  $k_{\theta^*}$  are the restrictions of  $k$  and  $k_*$  to  $\mathbb{R}^+ \times Y$ , as claimed.

Let us now denote the extended PS-realization with weighting pattern the restriction of  $k$  to  $\mathbb{R}^+ \times Y$  by

$$\theta_+ = (A_+, B_+, C_+; V_+, W_+, \tau_+; Y).$$

Let us construct the analogous extended PS-realization

$$\theta_- = (A_-, B_-, C_-; V_-, W_-, \tau_-; Y)$$

with weighting pattern the restriction of  $k$  to  $\mathbb{R}^- \times Y$  followed by the sign flip  $t \mapsto -t$ . We now define an extended PS-realization (7.41) with weighting pattern  $k$  as follows. Put

$$V = V_- \dot{+} V_+, \quad W = W_- \dot{+} W_+, \quad \tau = \begin{pmatrix} \tau_- & 0 \\ 0 & \tau_+ \end{pmatrix}, \quad (7.56a)$$

$$A = \begin{pmatrix} -A_- & 0 \\ 0 & A_+ \end{pmatrix}, \quad B = \begin{pmatrix} B_- \\ B_+ \end{pmatrix}, \quad C = (C_- \quad C_+), \quad (7.56b)$$

and arrange these data in an ordered 7-tuple  $\theta$  of the form (7.41). Then  $\theta$  is easily seen to be an extended PS-realization with weighting pattern  $k$ .  $\square$

Let us give the explicit form of an extended PS-realization with weighting pattern  $k$ . Define  $H$  as the bounded linear operator from  $L_\mu^2(\mathbb{R}; Y)$  into  $L_{-\mu}^2(\mathbb{R}; Y)$  by

$$(H\phi)(t) = \begin{cases} \int_0^\infty k(t+s, \phi(s)) ds, & t \in \mathbb{R}^+, \\ \int_{-\infty}^0 k(t+s, \phi(s)) ds, & t \in \mathbb{R}^-. \end{cases}$$

Let  $V$  be the closure of  $\text{Im } H$  in  $L_{-\mu}^2(\mathbb{R}; Y)$  and  $Q$  the orthogonal projection of  $L_\mu^2(\mathbb{R}; Y)$  along  $\text{Ker } H$ . We then let  $W$  stand for  $\text{Im } H$  endowed with the norm<sup>2</sup>

$$\|\phi\|_W = \left[ \|\phi\|_{L_{-\mu}^2(\mathbb{R}; Y)}^2 + \|Q\psi\|_{L_\mu^2(\mathbb{R}; Y)}^2 \right]^{1/2}, \quad \phi = H\psi.$$

Then we define  $B : Y \rightarrow V$  and  $C : W \rightarrow Y$  by

$$(Bu)(t) = k(t, u), \quad u \in Y, \quad t \in \mathbb{R} \text{ a.e.},$$

$$C\phi = \int_0^\infty k(t, (Q\psi)(t)) dt - \int_{-\infty}^0 k(t, (Q\psi)(t)) dt.$$

Putting

$$(E(t)\phi)(s) = \begin{cases} \text{sgn}(t)\phi(t+s), & ts > 0, \\ 0, & ts < 0, \end{cases}$$

$E(t; -iA)$  is defined as the restriction of  $E(t)$  to  $V$  and  $E(t; -iA_W)$  as the restriction of  $E(t)$  to  $W$ . Then

$$\theta = (A, B, C; V, W, \tau; Y)$$

with its constituent parts defined by (7.56) is an extended PS-realization with weighting pattern  $k$ .

If  $(t, u) \mapsto k(t, u)$  is the weighting pattern of a right PS-realization, then, according to Theorem 7.10,  $(t, u) \mapsto k_*(-t, u)$  is the weighting pattern of the “adjoint” left PS-realization. As a result, if  $(t, u) \mapsto k(t, u)$  is the weighting pattern of a right PS-realization, it necessarily satisfies the conditions of Theorem 7.17 and hence can be “realized” as the weighting pattern of an **extended** PS-realization. In other words, right PS-realizations and extended PS-realizations realize the same class of operator functions. Because Theorem 7.10 does not hold for left PS-realizations, no such realization result is known for left PS-realizations. Thus the realization problem for left PS-realizations is wide open.

## Chapter 8

# Mixed-type Functional Differential Equations

In this chapter we study linear functional differential equations of the form

$$x'(t) = \int_{-q}^p d\eta(\theta) x(t + \theta) + h(t), \quad (8.1)$$

where  $-q < 0 < p$ ,  $x(t) \in \mathbb{C}^M$ , and  $d\eta(\theta)$  is an  $M \times M$  matrix of finite (complex-valued) Lebesgue-Stieltjes measures on  $[-q, p]$ . Equation (8.1) is called of *mixed type* if the measure matrix  $d\eta(\theta)$  is supported on both of the subintervals  $[0, p]$  and  $[-q, 0]$ . As an initial condition we assume  $x(t)$  to be known for  $t \in [-q, p]$ :

$$x(t) = \varphi(t), \quad -q \leq t \leq p.$$

The special case studied most has the form

$$x'(t) = \sum_{j=1}^N A_j x(t + r_j) + h(t), \quad (8.2)$$

where  $\{r_1, \dots, r_N\}$  is a subset of  $[-q, p]$  consisting of discrete shifts and  $A_1, \dots, A_N$  are complex  $M \times M$  matrices. Here the measure matrix  $d\eta(\theta) = \sum_{j=1}^N \delta(\theta - r_j) A_j$  is discrete. Equations (8.1) and (8.2) are called *autonomous*, because  $d\eta(\theta)$  does not depend on  $t \in [-q, p]$ .

If the measure matrix  $d\eta(\theta)$  is supported on the subinterval  $[0, p]$  and hence (8.1) has the form

$$x'(t) = \int_0^p d\eta(\theta) x(t + \theta) + h(t), \quad (8.3)$$

we deal with a so-called *retarded functional differential equation* or *delay equation*. The theory of these delay equations is well understood and has been covered in

[81, 56, 165] (also [60, Sec. VI.6]). In [81] a synopsis is given of its many applications to control problems, population dynamics of a single species, predator-prey models, spread of diseases, nuclear reactor physics, transmission lines, etc. The mixed-type functional differential equation (8.1) has been studied less extensively, in spite of its applications to travelling waves in lattices, spatially nonlocal equations of convolution type, spatial discretizations of shock-wave problems, singularly perturbed time-delay problems, and optimal control theory (See [120] and references therein). Although many of the applications lead to nonlinear functional differential equations, linearization leads to (8.1) or its linear and nonautonomous natural generalization.

In Section 8.1, following [119], we construct a particular solution of (8.1) for any  $h \in L^p(\mathbb{R}; \mathbb{C}^M)$  ( $1 \leq p \leq \infty$ ). This solution is obtained from  $h$  by convolution with a Green's function matrix. As an ancillary result we prove the nonexistence of nontrivial bounded solutions of the homogeneous functional differential equation

$$x'(t) = \int_{-q}^p d\eta(\theta) x(t + \theta). \quad (8.4)$$

After some preliminary spectral analysis in Section 8.2, in Section 8.3 we prove the existence of a unique solution of (8.4) on the whole real axis, provided the solution is known in  $C[-q, p]$ . Following [120], the strategy is to prove the decomposition of the complex Banach space of initial data,

$$P \dot{+} Q = C([-q, p]; \mathbb{C}^M),$$

into two closed linear subspaces  $P$  and  $Q$  which contain the initial data of the solutions that are bounded on either  $(-\infty, p]$  or  $[q, \infty)$ , respectively. Here we assume the triviality of the, at most, finite-dimensional subspace of  $C([-q, p]; \mathbb{C}^M)$  of initial data of solutions having at most polynomial growth as  $t \rightarrow \pm\infty$ , a subspace often called the center manifold. It turns out that the solutions having their initial data in  $P$  are exponentially decaying as  $t \rightarrow -\infty$  and those having their initial data in  $Q$  are exponentially decaying as  $t \rightarrow +\infty$ . In Section 8.3 we in fact generate all of the solutions of (8.4) in the form

$$u(\tau, t) = \begin{cases} [E(\tau; A)\Pi_Q\varphi](t), & t > 0, \\ -[E(\tau; A)\Pi_P\varphi](t), & t < 0, \end{cases}$$

where  $A$  is an exponentially dichotomous operator on  $C([-q, p]; \mathbb{C}^M)$ ,  $\Pi_P$  and  $\Pi_Q$  are the complementary projections of  $C([-q, p]; \mathbb{C}^M)$  onto  $P$  and  $Q$ , respectively, and  $\varphi$  is the initial data. In other words, we project the initial data  $\varphi$  to arrive at  $\Pi_P\varphi \in P$  and  $\Pi_Q\varphi \in Q$  and extend the former to a bounded solution on  $(-\infty, p]$  and the latter to a bounded solution on  $[-q, \infty)$ . Either contribution to the solution can be extended from its half-line of natural definition to the full line at the expense of becoming unbounded. Moreover,  $A$  is the differentiation operator on a domain of functions satisfying (8.4), while the constituent semigroups

are translation semigroups. We also provide an alternative proof of exponential dichotomy based on Theorem 1.7. In Section 8.4 we consider the case of the delay equation (8.3), where the subspace  $P$  turns out to be finite-dimensional and the bisemigroup convertible into a hyperbolic semigroup.

The systematic study of functional differential equations of the form (8.1) was initiated by Rustichini: he derived the basic spectral properties and, under certain discreteness conditions on the measure  $\eta$ , the exponential dichotomy of the operator  $A$  in [137]. In [138] the case in which  $A$  has imaginary eigenvalues and hence a nontrivial center manifold exists has been taken into account. Mallet-Paret [119] has described the particular solutions of (8.1) by a Green's function formalism, also in the nonautonomous case where the measure  $\eta$  depends on  $t$ . A systematic study of exponential dichotomy in the autonomous and nonautonomous case has been made by Mallet-Paret and Verduyn Lunel [120]. Exponential dichotomy in the nonautonomous case has been studied by different methods by Härterich, Sandstede, and Scheel [84].

## 8.1 The Green's function matrix

Letting  $W^{1,p}(\mathbb{R}; \mathbb{C}^M)$  stand for the complex Banach space of all functions in  $L^p(\mathbb{R}; \mathbb{C}^M)$  whose distributional derivative belongs to  $L^p(\mathbb{R}; \mathbb{C}^M)$ , we derive the following theorem [119, Theorem 4.1] yielding a particular solution of the functional differential equation (8.1). We apply this result to prove that the homogeneous functional differential equation (8.4) does not have bounded solutions in  $t \in \mathbb{R}$ , unless they are identically zero. This result was obtained by Mallet-Paret [119] for the functional differential equation (8.2) and can in fact be proved in the same way for the more general equation (8.1). We study the homogeneous functional differential equation (8.4) in more detail in Sections 8.2 and 8.3.

Let us introduce the entire  $M \times M$  matrix function

$$\Delta(\lambda) = \lambda I_M - \int_{-q}^p e^{\lambda\theta} d\eta(\theta). \quad (8.5)$$

Then  $\lambda^{-1}\Delta(\lambda) = I_M + O(1/|\operatorname{Im} \lambda|)$  as  $|\operatorname{Im} \lambda| \rightarrow +\infty$ , uniformly in  $\operatorname{Re} \lambda$  on every vertical strip in the complex plane of finite width.

**Theorem 8.1.** *Let  $\det \Delta(\lambda) \neq 0$  for imaginary  $\lambda$ . Then for  $1 \leq p \leq \infty$  the linear operator  $\Lambda$  defined by*

$$(\Lambda x)(t) = x'(t) - \int_{-q}^p d\eta(\theta) x(t + \theta) \quad (8.6)$$

*is an invertible operator from  $W^{1,p}(\mathbb{R}; \mathbb{C}^M)$  onto  $L^p(\mathbb{R}; \mathbb{C}^M)$  and its inverse is given by*

$$(\Lambda^{-1}h)(t) = \int_{-\infty}^{\infty} G(t - \tau)h(\tau) d\tau.$$



Here the Green's function matrix  $G : \mathbb{R} \rightarrow \mathbb{C}^{M \times M}$  satisfies the exponential estimate

$$\|G(t)\| \leq Ce^{-\alpha|t|}, \quad 0 \neq t \in \mathbb{R},$$

for certain constants  $C, \alpha > 0$ . Moreover,  $\Lambda^{-1}h$  is a bounded continuous vector function for every  $h \in L^\infty(\mathbb{R}; \mathbb{C}^M)$ .

*Proof.* Since  $\lambda^{-1}\Delta(\lambda) = I_M + O(\lambda^{-1})$  as  $\lambda \rightarrow \infty$  along the imaginary axis, we have  $\lambda\Delta(\lambda)^{-1} = I_M + O(\lambda^{-1})$  as  $\lambda \rightarrow \infty$  along the imaginary axis and hence  $\zeta \mapsto \Delta(i\zeta)^{-1}$  belongs to  $L^2(\mathbb{R}; \mathbb{C}^{M \times M})$ . Write

$$\Delta(\lambda)^{-1} = \int_{-\infty}^{\infty} e^{-\lambda t} G(t) dt, \quad |\operatorname{Re} \lambda| \leq \varepsilon, \quad (8.7)$$

where  $G \in L^2(\mathbb{R}; \mathbb{C}^{M \times M})$  and  $\det \Delta(\lambda) \neq 0$  for  $|\operatorname{Re} \lambda| \leq \varepsilon$ . Since  $\Delta(\lambda)^{-1}$  is square integrable on each vertical line  $\lambda = \alpha + i\zeta$  ( $\zeta \in \mathbb{R}$ ) for  $\alpha \in [-\varepsilon, \varepsilon]$ , we have  $e^{\alpha|\cdot|}G(\cdot) \in L^2(\mathbb{R}; \mathbb{C}^{M \times M})$  for  $\alpha \in [0, \varepsilon]$ .

Since

$$\Delta(\lambda)^{-1} = \frac{1}{\lambda + 1} I_M + O(|\operatorname{Im} \lambda|^{-2}), \quad |\operatorname{Im} \lambda| \rightarrow +\infty,$$

uniformly in  $\operatorname{Re} \lambda \in [-\varepsilon, \varepsilon]$ , we have

$$G(t) = e^{-t} \chi_{\mathbb{R}^+}(t) + R(t),$$

where

$$\int_{-\infty}^{\infty} e^{-\lambda t} R(t) dt = O(|\operatorname{Im} \lambda|^{-2}), \quad |\operatorname{Im} \lambda| \rightarrow \infty,$$

uniformly in  $\operatorname{Re} \lambda \in [-\varepsilon, \varepsilon]$ . Thus for each  $\alpha \in [-\varepsilon, \varepsilon]$  the Fourier transforms of  $e^{-\alpha t} R(t)$  belong to  $L^1(\mathbb{R}; \mathbb{C}^{M \times M})$ . Hence,  $R(t)$  is continuous in  $t \in \mathbb{R}$  and  $e^{\varepsilon|t|}\|R(t)\| = o(1)$  as  $t \rightarrow \pm\infty$ . Consequently,  $G$  is continuous in  $0 \neq t \in \mathbb{R}$ , is exponentially decaying as  $t \rightarrow \pm\infty$ , and satisfies the jump condition

$$G(0^+) - G(0^-) = I_M. \quad (8.8)$$

As  $G \in L^1(\mathbb{R}; \mathbb{C}^{M \times M})$ , the convolution operator  $\Xi$  defined by

$$(\Xi h)(t) = \int_{-\infty}^{\infty} G(t - \tau) h(\tau) d\tau \quad (8.9)$$

is bounded on the Banach spaces  $L^p(\mathbb{R}; \mathbb{C}^M)$  ( $1 \leq p \leq \infty$ ) and  $BC(\mathbb{R}; \mathbb{C}^M)$ , in all of these cases with norm bounded above by  $\|G\|_1$ . As a result of (8.7), for  $p = 2$  the norm of  $\Xi$  coincides with  $\sup_{\operatorname{Re} \lambda = 0} \|\Delta(\lambda)^{-1}\|$ . Further,  $\Xi$  maps  $L^\infty(\mathbb{R}; \mathbb{C}^M)$  into  $BC(\mathbb{R}; \mathbb{C}^M)$ .

Let us now interpret  $G$  as a matrix of tempered distributions. We then define the matrix of tempered distributions

$$\Gamma(t) = G'(t) - \int_{-q}^p d\eta(\theta)G(t + \theta),$$

written as if it were a matrix function. For  $\operatorname{Re} \lambda = 0$  we then have, in the sense of tempered distributions,

$$\mathcal{L}[\Gamma](\lambda) = \left( \lambda I_M - \int_{-q}^p e^{\lambda\theta} d\eta(\theta) \right) \int_{-\infty}^{\infty} e^{-\lambda t} G(t) dt = I_M,$$

so that (cf. (8.8))

$$G'(t) = \delta(t)I_M + \int_{-q}^p d\eta(\theta)G(t + \theta),$$

where  $\delta$  denotes Dirac's delta function. Consequently (cf. (8.9)), we have in the distributional sense

$$(\Xi h)'(t) = h(t) + \int_{-\infty}^{\infty} \int_{-q}^p d\eta(\theta)G(t - \tau + \theta)h(\tau) d\tau.$$

Thus,  $h \mapsto (\Xi h)'$  is a bounded linear operator on  $L^p(\mathbb{R}; \mathbb{C}^M)$  ( $1 \leq p \leq \infty$ ) with norm bounded above by  $1 + \|G\|_1 V(\eta)$ . Therefore,  $\Xi$  is a bounded linear operator from  $L^p(\mathbb{R}; \mathbb{C}^M)$  into the Sobolev space  $W^{1,p}(\mathbb{R}; \mathbb{C}^M)$ .

We now compute for a.e.  $t \in \mathbb{R}$ ,

$$\begin{aligned} (\Lambda \Xi h)(t) &= (\Xi h)'(t) - \int_{-q}^p d\eta(\theta)(\Xi h)(t + \theta) \\ &= h(t) + \int_{-\infty}^{\infty} \int_{-q}^p d\eta(\theta)G(t - \tau + \theta)h(\tau) d\tau - \int_{-q}^p d\eta(\theta)(\Xi h)(t + \theta) \\ &= h(t). \end{aligned}$$

Consequently,  $\Xi = \Lambda^{-1}$ , as claimed.  $\square$

**Corollary 8.2.** *Let  $\det \Delta(\lambda) \neq 0$  for imaginary  $\lambda$ . Then the functional differential equation (8.4) does not have nontrivial solutions that are bounded on the full real line.*

## 8.2 Elementary spectral analysis

To write the homogeneous functional differential equation (8.4) as a linear autonomous differential equation on the complex Banach space  $X = C([-q, p]; \mathbb{C}^M)$ , we reformulate (8.4) as follows:

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial \theta}, \tag{8.10}$$

where the solution  $x(t, \theta)$  depends on  $t \in \mathbb{R}$  and  $\theta \in [-q, p]$  and satisfies the boundary condition

$$\left. \frac{\partial x}{\partial \theta} \right|_{\theta=0} = \int_{-q}^p d\eta(s)x(t, s).$$

We then obtain the differential equation

$$\frac{du}{dt} = Au(t),$$

where  $u : \mathbb{R} \rightarrow X$  is a vector function and  $A(X \rightarrow X)$  is defined by (8.11).

The following result is due to Rustichini [137].

**Theorem 8.3.** *Let us define the linear operator  $A$  on the complex Banach space  $X = C([-q, p]; \mathbb{C}^M)$  by*

$$\begin{cases} \mathcal{D}(A) = \left\{ \varphi \in C^1([-q, p]; \mathbb{C}^M) : \varphi'(0) = \int_{-q}^p d\eta(\theta) \varphi(\theta) \right\}, \\ A\varphi = \varphi' \text{ for } \varphi \in \mathcal{D}(A). \end{cases} \quad (8.11)$$

*Then  $A$  has a pure eigenvalue spectrum consisting of the zeros of the equation*

$$\det \Delta(\lambda) = 0,$$

*where  $\Delta(\lambda)$  is given by (8.5). Further,  $A$  has a compact resolvent. Moreover, each vertical strip  $\{\lambda \in \mathbb{C} : a \leq \operatorname{Re} \lambda \leq b\}$  of finite width contains at most finitely many eigenvalues of  $A$ .*

*Proof.* For  $\lambda \in \mathbb{C}$  and  $\psi \in X$  we consider the equation

$$(A - \lambda I_X)\phi = \psi,$$

where  $\phi \in \mathcal{D}(A)$ . Then for  $\theta \in [-q, p]$  we have

$$\frac{d}{d\theta} (e^{-\lambda\theta} \phi(\theta)) = e^{-\lambda\theta} \psi(\theta),$$

and therefore  $\phi \in C^1([-q, p]; \mathbb{C}^M)$  and

$$\phi(\theta) = e^{\lambda\theta} \phi(0) + \int_0^\theta e^{\lambda(\theta-\vartheta)} \psi(\vartheta) d\vartheta. \quad (8.12)$$

We now have to find  $\phi(0)$  such that  $\phi'(0) = \int_{-q}^p d\eta(\theta) \phi(\theta)$ . Substituting (8.12) into this condition we get

$$\Delta(\lambda) \phi(0) = -\psi(0) + \int_{-q}^p d\eta(\theta) \int_0^\theta e^{\lambda(\theta-\vartheta)} \psi(\vartheta) d\vartheta. \quad (8.13)$$

So the spectrum coincides with the set of zeros of  $\det \Delta(\cdot)$ . If  $\det \Delta(\lambda) = 0$ , then  $\lambda$  is an eigenvalue of  $A$  and the corresponding eigenspace is given by  $\{e^{\lambda(\cdot)}\xi : \Delta(\lambda)\xi = 0\}$ .

The resolvent of  $A$  is given by

$$((\lambda - A)^{-1}\psi)(\theta) = e^{\lambda\theta}\Delta(\lambda)^{-1}\left\{\psi(0) - \int_{-q}^p d\eta(\hat{\theta}) \int_0^{\hat{\theta}} e^{\lambda(\hat{\theta}-\vartheta)}\psi(\vartheta) d\vartheta\right\} - \int_0^{\theta} e^{\lambda(\theta-\vartheta)}\psi(\vartheta) d\vartheta, \quad (8.14)$$

where  $\theta \in [-q, p]$ . Thus  $A(X \rightarrow X)$  has a compact resolvent. In fact, the resolvent operator is the sum of a) an operator of finite rank and b) a direct sum of two Volterra integral operators, one on  $X_+ \stackrel{\text{def}}{=} C([0, p]; \mathbb{C}^M)$  and the other on  $X_- \stackrel{\text{def}}{=} C([-q, 0]; \mathbb{C}^M)$ .

Finally, observe that for  $a, b \in \mathbb{R}$  with  $a < b$ ,

$$\lambda^{-1}\Delta(\lambda) = I_M + o(1), \quad |\operatorname{Im} \lambda| \rightarrow +\infty,$$

uniformly in  $\operatorname{Re} \lambda$  in each vertical strip  $\{\lambda \in \mathbb{C} : a \leq \operatorname{Re} \lambda \leq b\}$ . Thus there are only finitely many zeros of  $\det \Delta(\lambda) = 0$  in each such strip.  $\square$

We now discuss an auxiliary linear operator  $A_0$  densely defined on the closed subspace  $X_0 = \{\varphi \in X : \varphi(0) = 0\}$  by

$$\begin{cases} \mathcal{D}(A_0) = \{\varphi \in C^1([-q, p]; \mathbb{C}^M) : \varphi'(0) = 0\}, \\ A_0\varphi = \varphi' \text{ for } \varphi \in \mathcal{D}(A_0). \end{cases} \quad (8.15)$$

Then  $A_0$  is a closed and densely defined linear operator on  $X_0$  with empty spectrum and its compact resolvent is given by

$$((\lambda - A_0)^{-1}\psi)(\theta) = - \int_0^{\theta} e^{\lambda(\theta-\vartheta)}\psi(\vartheta) d\vartheta.$$

Then  $(\lambda - A_0)^{-1}$  may (and will) be viewed as a bounded linear operator from  $X$  into  $X_0$ . We may then write

$$((\lambda - A_0)^{-1}\psi)(\theta) = \int_{-\infty}^{\infty} e^{-\lambda t} (E(t; A_0)\psi)(\theta) dt, \quad (8.16)$$

where

$$(E(t; A_0)\psi)(\theta) = \begin{cases} -\psi(t + \theta), & -p \leq -\theta < t < 0, \\ +\psi(t + \theta), & 0 < t < -\theta \leq q, \\ 0, & \text{otherwise.} \end{cases} \quad (8.17)$$

For  $0 \neq t \in \mathbb{R}$  the operators  $E(t; A_0)$  map  $X_0$  into  $X$  (and not necessarily into  $X_0$ ). For  $0 \neq t \in [-p, q]$  the function  $E(t; A_0)\psi$  has a jump discontinuity at

$\theta = -t$  whenever  $\psi \in X$  with  $\psi(0) \neq 0$ . On the contrary,  $E(t; A_0)$  is the zero operator if  $t \notin [-p, q]$ . Hence, for  $0 \neq t \in \mathbb{R}$  the operators  $E(t; A_0)$  map  $X$  into  $L^\infty([-q, p]; \mathbb{C}^M)$ . Nevertheless, in spite of the representation (8.16) and the decay of  $E(t; A_0)$  as  $t \rightarrow \pm\infty$ ,  $A_0(X_0 \rightarrow X_0)$  is not exponentially dichotomous.

Let us now discuss a special case of the operator  $A$ , where it is possible to perform all calculations explicitly.

**Example 8.4.** For a complex  $M \times M$  matrix  $\gamma$ , we define the operator  $A_\gamma(X \rightarrow X)$  by

$$\begin{cases} \mathcal{D}(A_\gamma) = \{\varphi \in C^1([-q, p]; \mathbb{C}^M) : \varphi'(0) = \gamma\varphi(0)\}, \\ A_\gamma\varphi = \varphi' \text{ for } \varphi \in \mathcal{D}(A_\gamma). \end{cases}$$

Then  $A_\gamma$  results from  $A(X \rightarrow X)$  by taking for  $\eta$  the complex measure concentrated at  $\theta = 0$  with weight  $\gamma$ , which implies that  $\Delta(\lambda) = \lambda I_M - \gamma$ . Thus the spectrum of  $A_\gamma$  coincides with that of the matrix  $\gamma$ . Its resolvent is compact and is given by

$$((\lambda - A_\gamma)^{-1}\psi)(\theta) = e^{\lambda\theta}(\lambda I_M - \gamma)^{-1}\psi(0) - \int_0^\theta e^{\lambda(\theta-\vartheta)}\psi(\vartheta) d\vartheta, \quad (8.18)$$

where  $\psi \in X$ . Although it will follow from Theorem 8.10 that  $A_\gamma$  is exponentially dichotomous iff  $\gamma$  does not have imaginary eigenvalues, we nevertheless present a direct proof.

Suppose  $\gamma$  does not have imaginary eigenvalues. Then

$$((\lambda - A_\gamma)^{-1}\psi)(\theta) = \int_{-\infty}^{\infty} e^{-\lambda t} [E(t + \theta; \gamma)\psi(0) + (E(t; A_0)\psi)(\theta)] dt,$$

where  $E(\cdot; \gamma)$  is the bisemigroup on  $\mathbb{C}^M$  generated by  $\gamma$  and  $(E(t; A_0)\psi)(\theta)$  is defined by (8.17). It is easily seen that  $E(\cdot; \gamma)$  coincides with the Green's function matrix defined by (8.7). Computing the one-sided limits of the expression  $E(t + \theta; \gamma)\psi(0) + (E(t; A_0)\psi)(\theta)$ , we obtain the following:

$\theta \rightarrow (-t)^+$ for $t > 0$ :	$E(0^+; \gamma)\psi(0)$
$\theta \rightarrow (-t)^-$ for $t > 0$ :	$E(0^-; \gamma)\psi(0) + \psi(0)$
$\theta \rightarrow (-t)^+$ for $t < 0$ :	$E(0^+; \gamma)\psi(0) - \psi(0)$
$\theta \rightarrow (-t)^-$ for $t < 0$ :	$E(0^-; \gamma)\psi(0)$

Thus, for  $0 \neq t \in \mathbb{R}$ ,  $E(t + \theta; \gamma)\psi(0) + (E(t; A_0)\psi)(\theta)$  is continuous in  $\theta \in [-q, p]$ . Using the exponential dichotomy of  $\gamma$  and the fact that  $E(t; A_0) = 0$  for  $t \notin [-p, q]$ , we obtain the estimate needed to apply Theorem 1.7 and to prove that  $A_\gamma$  is exponentially dichotomous, which concludes the example. When  $\gamma$  is the zero matrix, we obtain an extension of the operator  $A_0$  given by (8.15), but this extension has an eigenvalue in  $\lambda = 0$  and hence is not exponentially dichotomous.

We remark that the Jordan chains of  $A$  of length  $\sigma$  corresponding to the eigenvalue  $\lambda$  of  $A$  are exactly the chains of functions  $\phi_0, \phi_1, \dots, \phi_{\sigma-1}$  defined by  $\phi_0(\theta) = e^{\lambda\theta}\xi_0$  and

$$\phi_s(\theta) = e^{\lambda\theta} \sum_{t=0}^{s-1} \frac{\theta^t}{t!} \xi_{s-1-t}, \quad s = 1, 2, \dots, \sigma - 1,$$

where  $\xi_0 \neq 0$ ,  $\Delta(\lambda)\xi_0 = 0$ , and

$$\Delta(\lambda)\xi_s = -\xi_{s-1} + \sum_{t=0}^{s-1} \left( \int_{-q}^p \frac{\theta^{t+1}}{(t+1)!} e^{\lambda\theta} d\eta(\theta) \right) \xi_{s-1-t}, \quad s = 1, \dots, \sigma - 1.$$

Then it is easily verified that  $\{\phi_0, \phi_1, \dots, \phi_{\sigma-1}\} \subset \mathcal{D}(A)$ ,  $(A - \lambda)\phi_0 = 0$ , and  $(A - \lambda)\phi_s = \phi_{s-1}$  ( $s = 1, \dots, \sigma - 1$ ). When specifying the Jordan chains for the special case of Example 8.4 ( $\eta$  concentrated at  $\theta = 0$  with weight  $\gamma$ ), we get  $\phi_0(\theta) = e^{\lambda\theta}\xi_0$  and

$$\phi_s(\theta) = e^{\lambda\theta} \sum_{t=0}^{s-1} \frac{\theta^t}{t!} \xi_{s-1-t}, \quad s = 1, 2, \dots, \sigma - 1,$$

where  $\gamma\xi_0 = \lambda\xi_0$  and  $\gamma\xi_s = \lambda\xi_s + \xi_{s-1}$  ( $s = 1, \dots, \sigma - 1$ ).

The estimate

$$\|\Delta(\lambda)\| \leq |\lambda| + V(\eta) \max(e^{p\operatorname{Re} \lambda}, e^{q|\operatorname{Re} \lambda|})$$

implies that  $\Delta(\lambda)$  is an entire  $M \times M$  matrix function of order at most 1. Then  $\det \Delta(\lambda)$  is an entire function of order at most 1. Here we recall [117] that the order of an entire function  $f(\lambda)$  is given by

$$\limsup_{r \rightarrow \infty} \frac{\log \log \max_{|\lambda|=r} \|f(\lambda)\|}{\log r}.$$

When the measure  $\eta$  is discrete and not concentrated in  $\theta = 0$ , the order of  $\Delta(\lambda)$  is exactly 1.

Let us now estimate the resolvent of  $A$  (cf. [137]).

**Proposition 8.5.** *If  $\det \Delta(\lambda) \neq 0$ , we have*

$$\begin{aligned} & \|(\lambda - A)^{-1}\| \\ & \leq \begin{cases} \|\Delta(\lambda)^{-1}\| \left( 1 + \frac{e^{p\operatorname{Re} \lambda} - 1}{\operatorname{Re} \lambda} V(\eta) \right) + \frac{e^{p\operatorname{Re} \lambda} - 1}{\operatorname{Re} \lambda}, & \operatorname{Re} \lambda > 0, \\ \|\Delta(\lambda)^{-1}\| (1 + \max(p, q)V(\eta)) + \max(p, q), & \operatorname{Re} \lambda = 0, \\ \|\Delta(\lambda)^{-1}\| \left( 1 + \frac{e^{q|\operatorname{Re} \lambda|} - 1}{|\operatorname{Re} \lambda|} V(\eta) \right) + \frac{e^{q|\operatorname{Re} \lambda|} - 1}{|\operatorname{Re} \lambda|}, & \operatorname{Re} \lambda < 0, \end{cases} \end{aligned}$$

where  $V(\eta)$  is the total variation of the matrix-valued measure  $\eta$ .

*Proof.* Equation (8.12) implies that

$$\|\phi\|_X \leq \begin{cases} e^{p\operatorname{Re} \lambda} \|\phi(0)\| + \frac{e^{p\operatorname{Re} \lambda} - 1}{\operatorname{Re} \lambda} \|\psi\|_X, & \operatorname{Re} \lambda > 0, \\ \|\phi(0)\| + \max(p, q) \|\psi\|_X, & \operatorname{Re} \lambda = 0, \\ e^{q\operatorname{Re} \lambda} \|\phi(0)\| + \frac{e^{q|\operatorname{Re} \lambda|} - 1}{|\operatorname{Re} \lambda|} \|\psi\|_X, & \operatorname{Re} \lambda < 0. \end{cases}$$

On the other hand, (8.13) implies

$$\|\Delta(\lambda)\phi(0)\| \leq \begin{cases} \|\psi\|_X \left(1 + \frac{e^{p\operatorname{Re} \lambda} - 1}{\operatorname{Re} \lambda} V(\eta)\right), & \operatorname{Re} \lambda > 0, \\ \|\psi\|_X (1 + \max(p, q) V(\eta)), & \operatorname{Re} \lambda = 0, \\ \|\psi\|_X \left(1 + \frac{e^{q|\operatorname{Re} \lambda|} - 1}{|\operatorname{Re} \lambda|} V(\eta)\right), & \operatorname{Re} \lambda < 0, \end{cases}$$

where  $V(\eta)$  is the total variation of the matrix-valued measure  $\eta$ . These two estimates imply the proposition.  $\square$

Using (8.5) we easily derive

**Corollary 8.6.** *On each vertical strip  $\{\lambda \in \mathbb{C} : a \leq \operatorname{Re} \lambda \leq b\}$  of finite width we have*

$$\|(\lambda - A)^{-1}\| = O(|\operatorname{Im} \lambda|^{-1}), \quad |\operatorname{Im} \lambda| \rightarrow \infty,$$

*uniformly in  $\operatorname{Re} \lambda \in [a, b]$ .*

### 8.3 Exponential dichotomy

A necessary condition for  $A$  to be exponentially dichotomous on  $X = C([-q, p]; \mathbb{C}^M)$  is for  $A$  not to have any imaginary eigenvalues, i.e., for  $\det \Delta(\lambda) = 0$  not to have imaginary zeros. In that case there exists  $\varepsilon > 0$  such that  $\det \Delta(\lambda) = 0$  does not have any zeros with real part in  $[-\varepsilon, \varepsilon]$ , i.e., such that  $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq \varepsilon\}$  is contained in the resolvent set of  $A$ . Since  $A$  is not a Hilbert space and hence the conclusions of Theorem 1.9 may not hold, as yet we cannot draw the conclusion that  $A$  is exponentially dichotomous on  $X$  whenever  $\det \Delta(\lambda) = 0$  does not have any imaginary zeros. Nevertheless, this is exactly what we are going to prove in this section.

In this section we give two proofs of the exponential dichotomy of  $A$ . The first proof, given in [120], relies on a decomposition of the underlying Banach space into a subspace of initial conditions of solutions bounded on  $(-\infty, p]$  and a subspace of initial conditions of solutions bounded on  $[-q, \infty)$ . Once the decomposition is established, the proof is more or less straightforward, even though it relies heavily on the Ascoli-Arzelà Theorem 1.5. The second proof departs from the

representation (8.14) of the resolvent of  $A$  and involves applying Theorem 1.7 directly. In fact, it generalizes the procedure given in Example 8.4.

The first proof is very different from those given in the earlier chapters. Starting from the vector equation  $u'(t) = Au(t)$ ,  $t \in \mathbb{R}$ , in  $X$ , we define subspaces  $P$  and  $Q$  of initial values  $u(0)$  such that  $u(t)$  is bounded in  $X$  for  $t \in \mathbb{R}^-$  and  $u(0) \in P$ , and for  $t \in \mathbb{R}^+$  and  $u(0) \in Q$ . Then nontrivial solutions bounded for  $t \in \mathbb{R}$  are excluded by requiring that  $\det \Delta(\lambda) \neq 0$  for imaginary  $\lambda$ , which is tantamount to requiring that  $A$  does not have imaginary eigenvalues. A similar strategy would work if  $A$  is a square matrix without imaginary eigenvalues. The dissimilarity of the first proof with respect to anything presented in the earlier chapters has been the primary reason to include it, in spite of its length.

### 8.3.1 Decomposing the underlying Banach space

Let us associate with the mixed-type functional differential equation (8.4) the following two complex vector spaces:

$$\begin{aligned} \mathcal{P} &= \{x : (-\infty, p] \rightarrow \mathbb{C}^M : x \text{ is a bounded solution of (8.1) on } (-\infty, 0]\}; \\ \mathcal{Q} &= \{y : [-q, \infty) \rightarrow \mathbb{C}^M : y \text{ is a bounded solution of (8.1) on } [0, \infty)\}. \end{aligned}$$

We write  $P$  and  $Q$  for the linear vector spaces consisting of their initial conditions. More precisely,

$$\begin{aligned} P &= \{\varphi \in C([-q, p]; \mathbb{C}^M) : \varphi = x|_{[-q, p]} \text{ for some } x \in \mathcal{P}\}; \\ Q &= \{\varphi \in C([-q, p]; \mathbb{C}^M) : \varphi = y|_{[-q, p]} \text{ for some } y \in \mathcal{Q}\}. \end{aligned}$$

We then call  $x \in \mathcal{P}$  a *left prolongation* of  $\varphi \in P$  and  $y \in \mathcal{Q}$  a *right prolongation* of  $\varphi \in Q$ . Clearly, for each  $t \in \mathbb{R}^-$  the translations  $s \mapsto x(s+t)$  belong to  $\mathcal{P}$  if  $x \in \mathcal{P}$ , and for every  $t \in \mathbb{R}^+$  the translations  $s \mapsto y(s+t)$  belong to  $\mathcal{Q}$  if  $y \in \mathcal{Q}$ .

**Proposition 8.7.** *Every  $\varphi \in P$  has a unique left prolongation and every  $\psi \in Q$  has a unique right prolongation. Further,  $P \cap Q = \{0\}$ .*

*Proof.* It is sufficient to prove that the zero solution is the only left prolongation of the zero element of  $P$ . Indeed, if this were not the case, then by extending such an  $x \in \mathcal{P}$  to the full real line by defining  $x(t) = 0$  for  $t > p$ , we would obtain a nontrivial bounded solution of (8.4) on  $\mathbb{R}$ , which contradicts Theorem 8.1. The same theorem implies that  $\mathcal{P} \cap \mathcal{Q} = \{0\}$  and hence  $P \cap Q = \{0\}$ .  $\square$

The following result is due to Rustichini [137, 138]. Here we derive it from the crucial Proposition 8.7, using arguments given in [120]. Similar results have been derived by different means in [84].



**Proposition 8.8.** *Suppose  $\det \Delta(\lambda) \neq 0$  for imaginary  $\lambda$ . Then there exist constants  $C, \alpha > 0$  such that*

$$\|x(t)\| + \|x'(t)\| \leq Ce^{\alpha t} \|x_0\|, \quad t \in \mathbb{R}^-, \quad (8.19a)$$

$$\|y(t)\| + \|y'(t)\| \leq Ce^{-\alpha t} \|y_0\|, \quad t \in \mathbb{R}^+, \quad (8.19b)$$

for every  $x_0 \in \mathcal{P}$  and  $y_0 \in \mathcal{Q}$ .

*Proof.* **1.** We first show the existence of  $\tau > -p$  such that for every  $x \in \mathcal{P}$

$$\|x(t)\| \leq \frac{1}{2} \sup_{s \leq p} \|x(s)\|, \quad t \leq -\tau. \quad (8.20)$$

Indeed, if (8.20) were not true, there would exist sequences  $\{\tau_n\}_{n=1}^\infty$  with  $\tau_n \rightarrow +\infty$  and  $\{x_n\}_{n=1}^\infty$  in  $\mathcal{P}$  such that for  $n = 1, 2, 3, \dots$

$$\|x_n(-\tau_n)\| > \frac{1}{2}, \quad \sup_{s \leq p} \|x_n(s)\| = 1.$$

Put  $z_n(t) = x_n(t - \tau_n)$ . Then  $z_n$  satisfies (8.4) in the interval  $(-\infty, \tau_n]$ . On every compact subinterval of  $\mathbb{R}$ , the sequence  $\{z_n\}_{n=1}^\infty$  is uniformly bounded and equicontinuous. Thus for some subsequence we have the limit  $z_{n_k}(t) \rightarrow z(t)$  uniformly in  $t$  on compact intervals, by Theorem 1.5. Further, using that

$$z_n(t_2) - z_n(t_1) = \int_{t_1}^{t_2} \int_{-q}^p d\eta(\theta) z_n(s + \theta) ds,$$

we see that  $z$  is a solution of (8.4) for  $t \in \mathbb{R}$ . This solution  $z$  is bounded on  $\mathbb{R}$  and nontrivial, since it satisfies  $\|z(0)\| \geq \frac{1}{2}$ . This contradicts Corollary 8.2. Hence, (8.20) is true for each  $x \in \mathcal{P}$ .

**2.** We now prove the existence of a constant  $K$  such that

$$\|x(t)\| \leq K \|x_0\|, \quad t \leq p, \quad (8.21)$$

for each  $x \in \mathcal{P}$ , where  $x$  is the left prolongation of  $x_0 \in P$ . Indeed, if (8.21) were false for every constant  $K$ , then there would exist sequences  $\{x_n\}_{n=1}^\infty$  in  $\mathcal{P}$  and  $\{K_n\}_{n=1}^\infty$  with  $K_n \rightarrow +\infty$  such that

$$\sup_{t \leq p} \|x_n(t)\| = K_n \|x_{n0}\| = 1,$$

where  $x_n$  is the left prolongation of  $x_{n0} \in P$ . Let  $\|x_n(\cdot)\|$  attain its maximum in  $t = -\tau_n$ . Then, according to (8.20), we have  $-\tau \leq -\tau_n \leq -q$  and hence  $\{\tau_n\}_{n=1}^\infty$  is a bounded sequence in the compact interval  $[q, \tau]$ . Then, by the Bolzano-Weierstrass theorem [136], there exists a subsequence  $\{\tau_{n_k}\}_{k=1}^\infty$  which converges to  $\tau_0 \in [q, \tau]$ . Applying Theorem 1.5 and restricting ourselves to another subsequence,

we have  $x_{n_k}(t) \rightarrow x(t)$  uniformly in  $t$  on compact subsets of  $(-\infty, p]$ . Letting  $[-q, p]$  be one such compact subset, we see that  $\{x_{n_k,0}\}_{k=1}^\infty$  converges in the norm of  $C([-q, p]; \mathbb{C}^M)$ . Because of  $\|x_{n0}\| = (1/K_n) \rightarrow 0$ , we see that the limit function  $x(t)$  is a nontrivial (because  $\|x(-\tau_0)\| = 1$ ) left prolongation of the zero element of  $P$ , which contradicts Proposition 8.7. Consequently, (8.21) is true for each  $x \in \mathcal{P}$ .

3. The estimates (8.20) and (8.21) imply

$$\|x(t)\| \leq \frac{K}{2} \|x_0\|, \quad t \leq -\tau,$$

where  $x \in \mathcal{P}$  is the left prolongation of  $x_0 \in P$ . Putting, for  $m = 0, 1, 2, \dots$ ,  $x^{(m)}(t) = x(t - m(\tau + p))$  and letting  $x^{(m)} \in \mathcal{P}$  be the left prolongation of  $x_0^{(m)} \in P$ , we obtain, for  $m = 1, 2, \dots$ ,

$$\begin{aligned} \sup_{s \leq -\tau} \|x^{(m)}(s)\| &\leq \frac{1}{2} \sup_{s \leq p} \|x^{(m)}(s)\| = \frac{1}{2} \sup_{s \leq p} \|x^{(m-1)}(s - \tau - p)\| \\ &= \frac{1}{2} \sup_{s \leq -\tau} \|x^{(m-1)}(s)\| \leq \dots \leq \frac{1}{2^m} \sup_{s \leq -\tau} \|x(s)\| \\ &\leq \frac{K}{2^{m+1}} \|x_0\|, \end{aligned}$$

which implies that

$$\|x(t)\| \leq \frac{K}{2^{m+1}} \|x_0\|, \quad t \leq -\tau - m(\tau + p).$$

Hence, there exists  $\alpha > 0$  such that

$$\|x(t)\| \leq C e^{\alpha t} \|x_0\|, \quad t \in \mathbb{R}^-,$$

for each  $x \in \mathcal{P}$ . In fact, we may choose the constants as  $\alpha = (\ln 2)/(\tau + p)$  and  $C = K 2^{\tau/(\tau+p)}$ . The estimate (8.19a) then follows with the help of (8.4). The estimate (8.19b) is proved likewise.  $\square$

**Corollary 8.9.** *Suppose  $\det \Delta(\lambda) \neq 0$  for imaginary  $\lambda$ . Then  $P$  and  $Q$  are closed linear subspaces of  $C([-q, p]; \mathbb{C}^M)$ .*

*Proof.* Put  $X = C([-q, p]; \mathbb{C}^M)$ ,  $X^+ = C([0, p]; \mathbb{C}^M)$ , and  $X^- = C([-q, 0]; \mathbb{C}^M)$ . Define  $\pi^+ : X \rightarrow X^+$  and  $\pi^- : X \rightarrow X^-$  as the natural restriction operators. Suppose  $\{\varphi_n\}_{n=1}^\infty$  is a sequence in  $P$  such that  $\|\pi^+ \varphi_n - \psi\|_{X^+} \rightarrow 0$  for some  $\psi \in X^+$ . Then the sequence  $\{x_n\}_{n=1}^\infty$  of left prolongations of  $\varphi_n$  is uniformly bounded and equicontinuous in  $(-\infty, 0]$ . Thus, by Theorem 1.5, some subsequence  $\{x_{n_k}\}_{k=1}^\infty$  converges uniformly on compact subsets of  $(-\infty, 0]$ . Together with its uniform convergence on  $[0, p]$ , we get its uniform convergence on compact subsets of  $(-\infty, p]$ . Thus  $\{\varphi_{n_k}\}_{k=1}^\infty$  converges in  $X$  to some  $\varphi$  satisfying  $\pi^+ \varphi = \psi$ . Consequently,  $P$  is closed in  $X$ . In the same way we prove that  $Q$  is closed in  $X$ .  $\square$

We need two more propositions before stating the main result.

**Proposition 8.10.** *Suppose  $\det \Delta(\lambda) \neq 0$  for imaginary  $\lambda$ . Then the restriction operators  $\pi_P^- : P \rightarrow C([-q, 0]; \mathbb{C}^M)$  and  $\pi_Q^+ : Q \rightarrow C([0, p]; \mathbb{C}^M)$  are compact.*

*Proof.* The estimates (8.19) imply that the restrictions  $\pi_P^- \varphi$  to  $[-q, 0]$  of left prolongations  $x \in P$  are equicontinuous and uniformly bounded for  $\varphi \in P$  bounded, and similarly for the restrictions  $\pi_Q^+ \psi$  to  $[0, p]$  of right prolongations in  $Q$ . The compactness of these restriction operators then follows with the help of Theorem 1.5.  $\square$

We now apply this lemma to prove the following

**Proposition 8.11.** *Suppose  $\det \Delta(\lambda) \neq 0$  for imaginary  $\lambda$ . Then  $P$  and  $Q$  are closed linear subspaces of  $C([-q, p]; \mathbb{C}^M)$  and*

$$P + Q = C([-q, p]; \mathbb{C}^M). \quad (8.22)$$

*Proof.* We already know that  $P$  and  $Q$  are closed linear subspaces of

$$X = C([-q, p]; \mathbb{C}^M)$$

such that  $P \cap Q = \{0\}$ . Thus it suffices to prove that (1)  $P + Q$  is closed in  $X$ , and (2)  $P + Q$  is dense in  $X$ .

Indeed, let  $\{\varphi_n\}_{n=1}^\infty$  be a sequence in  $P$  and  $\{\psi_n\}_{n=1}^\infty$  a sequence in  $Q$  such that  $\rho_n = \varphi_n + \psi_n$  satisfies  $\|\rho_n - \rho\|_X \rightarrow 0$ . If  $\{\varphi_n\}_{n=1}^\infty$  and hence  $\{\psi_n\}_{n=1}^\infty$  were to be bounded sequences in  $X$ , then, according to Lemma 8.10, some subsequence of the restrictions  $\{\pi_P^- \varphi_n\}_{n=1}^\infty$  converges uniformly on  $[-q, 0]$ , and hence the corresponding subsequence of functions  $\pi^- \psi_n = \pi^- (\rho_n - \varphi_n)$  converges uniformly on  $[-q, 0]$ . By Lemma 8.10, a further subsequence of  $\{\pi_Q^+ \psi_n\}_{n=1}^\infty$  converges uniformly on  $[0, p]$ . Hence, a still further subsequence of  $\{\pi^+ \varphi_n\}_{n=1}^\infty$  converges uniformly on  $[0, p]$ . As a result, there exist  $\varphi, \psi \in X$  such that

$$\|\varphi_{n_k} - \varphi\|_X + \|\psi_{n_k} - \psi\|_X \rightarrow 0.$$

Since  $P$  and  $Q$  are closed, we have  $\varphi \in P$  and  $\psi \in Q$ . Thus  $\rho = \varphi + \psi \in P + Q$ .

Let us now suppose that the sequences  $\{\varphi_n\}_{n=1}^\infty$  in  $P$  and  $\{\psi_n\}_{n=1}^\infty$  in  $Q$  satisfying  $\rho_n = \varphi_n + \psi_n$  and  $\|\rho_n - \rho\|_X \rightarrow 0$  are unbounded in  $X$ , and let  $\kappa_n = \|\varphi_n\|_X + \|\psi_n\|_X \rightarrow +\infty$ . Put  $\tilde{\varphi}_n = \varphi_n/\kappa_n$ ,  $\tilde{\psi}_n = \psi_n/\kappa_n$ , and  $\tilde{\rho}_n = \rho_n/\kappa_n$ , which obviously are bounded sequences. Applying the argument of the preceding paragraph, we find  $\tilde{\varphi} \in P$  and  $\tilde{\psi} \in Q$  occurring as the limits of certain subsequences of  $\{\tilde{\varphi}_n\}_{n=1}^\infty$  and  $\{\tilde{\psi}_n\}_{n=1}^\infty$ . Then  $\|\tilde{\varphi}\|_X + \|\tilde{\psi}\|_X = 1$ , while  $\|\tilde{\rho}_n\|_X \rightarrow 0$  yields  $\tilde{\varphi} + \tilde{\psi} = 0$ . This would result in a nontrivial element  $\tilde{\varphi} = -\tilde{\psi}$  of  $P \cap Q$ , which is a contradiction. Hence this situation does not occur. We may thus conclude that  $P + Q$  is closed in  $X$ .

We now prove that  $C^1([-q, p]; \mathbb{C}^M) \subset P + Q$ , therewith proving the density of  $P + Q$  in  $X$ . Indeed, let  $\varphi \in C^1([-q, p]; \mathbb{C}^M)$  and extend it to some bounded

$C^1$ -function  $x : \mathbb{R} \rightarrow \mathbb{C}^M$  with bounded derivative. Then  $x \in W^{1,\infty}(\mathbb{R}; \mathbb{C}^M)$  with  $x_0 = \varphi$ . Put  $h = \Lambda x$ , where  $\Lambda$  is given by (8.6). Then, by Theorem 8.1,  $h \in L^\infty(\mathbb{R})$ . Put

$$h^-(t) = \begin{cases} h(t), & t \leq 0, \\ 0, & t > 0, \end{cases} \quad h^+(t) = \begin{cases} 0, & t \leq 0, \\ h(t), & t > 0. \end{cases}$$

Then Theorem 8.1 implies that  $x^\pm = \Lambda^{-1}h^\pm \in W^{1,\infty}(\mathbb{R}; \mathbb{C}^M)$ , while  $x = x^+ + x^-$ . Thus  $x^+ \in \mathcal{P}$  and  $x^- \in \mathcal{Q}$ , implying  $x_0^+ \in P$  and  $x_0^- \in Q$  with  $\varphi = x_0 = x_0^+ + x_0^- \in P + Q$ . Consequently,  $C^1([-q, p]; \mathbb{C}^M) \subset P + Q$ , as claimed.  $\square$

We now state and prove the main result of this chapter. Various versions of this result appeared in [137, 84, 120].

**Theorem 8.12.** *Suppose  $\det \Delta(\lambda) \neq 0$  for imaginary  $\lambda$ . Then the linear operator  $A$  defined by (8.11) is exponentially dichotomous.*

*Proof.* Above we have proved the direct sum decomposition (8.22) into the closed subspaces  $P$  and  $Q$  such that each  $x_0 \in P$  has a unique left prolongation  $x \in \mathcal{P}$ , each  $y_0 \in Q$  has a unique right prolongation  $y \in \mathcal{Q}$ ,  $x \in \mathcal{P}$  is exponentially decaying on  $\mathbb{R}^-$ , and  $y \in \mathcal{Q}$  is exponentially decaying on the right half-line. Letting  $x_t$  stand for the translated vector function  $x_t(\tau) = x(t + \tau)$ , we now put  $\rho = x_0 + y_0$  and define

$$E(t)\rho = \begin{cases} y_t, & t > 0, \\ -x_t, & t < 0. \end{cases}$$

Then  $E(t)$  is a bounded linear operator on  $X$  (also for  $t = 0^\pm$ ) and its norm is exponentially decaying as  $t \rightarrow \pm\infty$ . Further,

$$E(0^+) - E(0^-) = I_X.$$

Since  $x_t \in \mathcal{P}$  for  $x \in \mathcal{P}$  and  $t < 0$  and  $y_t \in \mathcal{Q}$  for  $y \in \mathcal{Q}$  and  $t > 0$ , we have the bisemigroup properties:

$$E(t + \tau) = \begin{cases} E(t)E(\tau), & t, \tau > 0, \\ -E(t)E(\tau), & t, \tau < 0, \\ 0, & t\tau < 0. \end{cases}$$

Thus  $\{E(t)\}_{0 \neq t \in \mathbb{R}}$  is a strongly continuous bisemigroup on  $X$ .

To find its infinitesimal generator, we compute

$$\begin{aligned} \frac{1}{t} [E(t) - E(0^+)] \rho &= \frac{y_t - y_0}{t}, & t > 0, \\ \frac{1}{t} [E(t) - E(0^-)] \rho &= \frac{x_t - x_0}{t}, & t < 0, \end{aligned}$$

which converge uniformly on  $[-q, p]$  to  $y'_0$  as  $t \rightarrow 0^+$  and to  $x'_0$  as  $t \rightarrow 0^-$ , respectively. Because of (8.4) we have

$$x'(0) = \int_{-q}^p d\eta(\theta)x(\theta), \quad y'(0) = \int_{-q}^p d\eta(\theta)y(\theta).$$

Hence  $x_0, y_0 \in \mathcal{D}(A)$ ,  $Ax_0 = x'$ , and  $Ay_0 = y'$ , where  $A$  is given by (8.11). Consequently,  $\rho \in \mathcal{D}(A)$  and  $A\rho = x' + y'$ ,  $\rho$  being the restriction of  $x + y$  to  $[-q, p]$ . We have thus proved that  $A$  is the infinitesimal generator of the above bisemigroup.  $\square$

Let us consider the natural restriction operators

$$\begin{aligned} \pi_P^- : P &\rightarrow C([-q, 0]; \mathbb{C}^M), & \pi_Q^+ : Q &\rightarrow C([0, p]; \mathbb{C}^M), \\ \pi_P^+ : P &\rightarrow C([0, p]; \mathbb{C}^M), & \pi_Q^- : Q &\rightarrow C([-q, 0]; \mathbb{C}^M). \end{aligned}$$

Denote by  $\Pi_P$  and  $\Pi_Q$  the complementary projections on  $C([-q, p]; \mathbb{C}^M)$  with ranges  $P$  and  $Q$ . Introduce

$$Vx_0 = (\pi_P^+\varphi) \dot{+} (\pi_Q^-\psi), \quad Wx_0 = (\pi_P^-\varphi) \dot{+} (\pi_Q^+\psi),$$

where  $x_0 = \varphi + \psi$  with  $\varphi \in P$  and  $\psi \in Q$ . Then  $V$  and  $W$  map  $C([-q, p]; \mathbb{C}^M)$  into  $C([-q, 0]; \mathbb{C}^M) \dot{+} C([0, p]; \mathbb{C}^M)$ , a space which can be viewed as the space of continuous functions  $[-q, p] \mapsto \mathbb{C}^M$  with a jump discontinuity in zero. Clearly, this space contains  $C([-q, p]; \mathbb{C}^M)$  as a closed complemented subspace of codimension  $M$ . Further, by Proposition 8.10,  $W$  is a compact operator. Since  $V + W$  is the natural imbedding of  $C([-q, p]; \mathbb{C}^M)$  into  $C([-q, 0]; \mathbb{C}^M) \dot{+} C([0, p]; \mathbb{C}^M)$  and as such is a Fredholm operator of index  $-M$ , we see that  $\pi_P^+$  and  $\pi_Q^-$  are Fredholm operators whose Fredholm indices add up to  $-M$  (see [144] for Fredholm theory between distinct Banach spaces), thus reproducing Theorem 3.4 of [120].

The index property for  $\pi_P^+$  and  $\pi_Q^-$  implies that at least one of the following two types of solution of (8.4) must exist: a) nontrivial solutions that are bounded on  $\mathbb{R}^-$  and vanish on  $[0, p]$ , or b) nontrivial solutions that are bounded on  $\mathbb{R}^+$  and vanish on  $[-q, 0]$ .

### 8.3.2 Taking the inverse Laplace transform of the resolvent

We now give the second proof of the exponential dichotomy of  $A$ , based on Theorem 1.7.

In analogy with (8.18) we can write the resolvent (8.14) of  $A(X \rightarrow X)$  in the form

$$((\lambda - A)^{-1}\psi)(\theta) = \int_{-\infty}^{\infty} e^{-\lambda t} (E(t; A)\psi)(\theta) dt,$$

where (cf. (8.7) and (8.17))

$$\begin{aligned} (E(t; A)\psi)(\theta) &= G(t + \theta)\psi(0) + (E(t; A_0)\psi)(\theta) \\ &\quad + \int_{-\infty}^{\infty} G(t - s + \theta) \int_{-q}^p d\eta(\hat{\theta})(E(s; A_0)\psi)(\hat{\theta}) ds. \end{aligned} \quad (8.23)$$

Here we have relied on the analogy between the second and third terms in the right-hand side of (8.14) to arrive at the final term in the right-hand side of (8.23). For  $0 \neq t \in \mathbb{R}$ ,  $E(t; A_0)\psi$  is considered as belonging to  $L^\infty([-q, p]; \mathbb{C}^M)$  whenever  $\psi \in X$ .

Computing the one-sided limits of the expression

$$G(t + \theta)\psi(0) + (E(t; A_0)\psi)(\theta),$$

we obtain the following results.

$\theta \rightarrow (-t)^+$ for $t > 0$ :	$G(0^+)\psi(0)$
$\theta \rightarrow (-t)^-$ for $t > 0$ :	$G(0^-)\psi(0) + \psi(0)$
$\theta \rightarrow (-t)^+$ for $t < 0$ :	$G(0^+)\psi(0) - \psi(0)$
$\theta \rightarrow (-t)^-$ for $t < 0$ :	$G(0^-)\psi(0)$

Since

$$G(0^+) - G(0^-) = I_M,$$

it is now clear that  $E(t; A)(\theta)$  is continuous in  $\theta \in [-q, p]$  for each  $0 \neq t \in \mathbb{R}$ . In other words, for  $0 \neq t \in \mathbb{R}$  the first two terms on the right-hand side of (8.23) define a bounded linear operator from  $X$  into  $X$  (with norm bounded above by  $\|G\|_\infty + 1$ ).

Let us now analyze the third term on the right-hand side of (8.23). We first note that the vector function  $\int_{-q}^p d\eta(\hat{\theta})(E(\cdot; A_0)\psi)(\hat{\theta})$  belongs to  $L^\infty(\mathbb{R}; \mathbb{C}^M)$  and has compact support. Because the Green's function matrix  $G$  belongs to  $L^1(\mathbb{R}; \mathbb{C}^{M \times M})$ , is applied as a convolution to a bounded measurable vector function, and the result is then translated by  $\theta$ , the integral term in (8.23) is an expression that is continuous in  $\theta \in [-q, p]$  and  $t \in \mathbb{R}$ . Moreover, its  $L^\infty$  norm is bounded above by  $\|G\|_1 V(\eta)\|\psi\|_X$ .

Now recall that

$$\|G(t)\| \leq C e^{-\alpha|t|}, \quad 0 \neq t \in \mathbb{R}.$$

Then

$$\operatorname{ess\,sup}_{\theta \in [-q, p]} \|G(t + \theta)\| \leq C e^{\alpha \max(p, q)} e^{-\alpha|t|}.$$

Moreover, since  $E(t; A_0)$  is the zero operator for  $|t| > \max(p, q)$ , we easily see from (8.23) that, for  $|t| > \max(p, q)$ ,

$$\|(E(t; A)\psi)(\theta)\| \leq C e^{\alpha \max(p, q)} e^{-\alpha|t|} \|\psi\|_X \{1 + \operatorname{Var}(\eta)\}.$$

In other words, for  $0 \neq t \in \mathbb{R}$  and for  $t = 0^\pm$  the operator  $E(t; A)$  maps  $X$  into itself, while

$$\|E(t; A)\psi\|_X \leq C e^{\alpha \max(p, q)} e^{-\alpha|t|} \|\psi\|_X \{1 + \text{Var}(\eta)\}$$

for  $\psi \in X$  and  $|t| > \max(p, q)$ . Theorem 1.7 then implies that  $A(X \rightarrow X)$  is exponentially dichotomous whenever  $\det \Delta(\lambda) = 0$  does not have imaginary zeros. We have thus arrived at an alternative proof of Theorem 8.12.

## 8.4 Delay equations and hyperbolic semigroups

In this section we consider the retarded functional differential equation or delay equation

$$x'(t) = \int_{-q}^0 d\eta(\theta) x(t + \theta) + h(t), \quad (8.24)$$

where  $q > 0$ ,  $x(t) \in \mathbb{C}^M$ , and  $d\eta(\theta)$  is an  $M \times M$  matrix of finite (complex-valued) Stieltjes measures on  $[-q, 0]$ . Introducing the complex Banach space  $X_- = C([-q, 0]; \mathbb{C}^M)$ , we write (8.24) in the form (8.10), where the solution  $x(t, \theta)$  depends on  $t \in \mathbb{R}^+$  and  $\theta \in [-q, 0]$  and satisfies the boundary condition

$$\left. \frac{\partial x}{\partial \theta} \right|_{\theta=0} = \int_{-q}^0 d\eta(s) x(t, s) + h(t).$$

We then obtain the Cauchy problem

$$\frac{du}{dt} = A_- u(t), \quad u(0) = \varphi \in C([-q, 0]; \mathbb{C}^M).$$

The following result is well known [56, 60].

**Theorem 8.13.** *The linear operator  $A_-$  defined on the complex Banach space  $X_- = C([-q, 0]; \mathbb{C}^M)$  by*

$$\begin{cases} \mathcal{D}(A_-) = \left\{ \varphi \in C^1([-q, 0]; \mathbb{C}^M) : \varphi'(0) = \int_{-q}^0 d\eta(\theta) \varphi(\theta) \right\}, \\ A_- \varphi = \varphi' \text{ for } \varphi \in \mathcal{D}(A_-), \end{cases}$$

*has a pure eigenvalue spectrum consisting of the zeros of the determinant of the matrix function*

$$\Delta_-(\lambda) = \lambda I_M - \int_{-q}^0 e^{\lambda \theta} d\eta(\theta),$$

*has a compact resolvent, and has all of its eigenvalues concentrated in the half-plane  $\{\lambda \in \mathbb{C} : \text{Re } \lambda < \sigma\}$  for some  $\sigma \in \mathbb{R}$ . The operator  $A_-$  is the infinitesimal generator of a hyperbolic semigroup iff  $\det \Delta_-(\lambda) = 0$  does not have any imaginary zeros.*

*Proof.* Most of the statements of Theorem 8.13 follow from Theorems 8.3 and 8.12, except for the eigenvalues being concentrated in a left half-plane. Since each vertical strip of finite width only contains finitely many eigenvalues, it is clear that, under the condition that  $\det \Delta_-(\lambda) \neq 0$  for imaginary  $\lambda$ ,  $E(0^+; A_-)$  has finite rank and the restriction of  $A_-$  to its range is nonsingular. Thus under this condition,  $A_-$  is the infinitesimal generator of a strongly continuous semigroup on  $X_-$ .

It remains to prove that the eigenvalues of  $A_-$  are concentrated in a left half-plane. Indeed, the boundedness of the integral  $\int_{-q}^0 e^{\lambda\theta} d\eta(\theta)$  for  $\operatorname{Re} \lambda \geq 0$  implies that

$$\left\| I_M - \frac{1}{\lambda} \Delta_-(\lambda) \right\| \leq \frac{V(\eta)}{|\lambda|}, \quad \operatorname{Re} \lambda > 0 \text{ and } \lambda \neq 0.$$

Thus if  $|\lambda| > V(\eta)$  and  $\operatorname{Re} \lambda \geq 0$ , we have  $\det \Delta_-(\lambda) \neq 0$ , which proves the statement.  $\square$

In the same way we consider the “negative delay” equation

$$x'(t) = \int_0^p d\eta(\theta) x(t + \theta) + h(t), \quad (8.25)$$

where  $p > 0$ ,  $x(t) \in \mathbb{C}^M$ , and  $d\eta(\theta)$  is an  $M \times M$  matrix of finite (complex-valued) Stieltjes measures on  $[0, p]$ . Introducing the complex Banach space  $X_+ = C([0, p]; \mathbb{C}^M)$ , we write (8.25) in the form (8.10), where the solution  $x(t, \theta)$  depends on  $t \in \mathbb{R}^-$  and  $\theta \in [0, p]$  and satisfies the boundary condition

$$\left. \frac{\partial x}{\partial \theta} \right|_{\theta=0} = \int_0^p d\eta(s) x(t, s) + h(t).$$

We then obtain the Cauchy problem

$$\frac{du}{dt} = A_+ u(t), \quad u(0) = \varphi \in C([0, p]; \mathbb{C}^M).$$

Instead of Theorem 8.13, we derive the following result.

**Theorem 8.14.** *The linear operator  $A_+$  defined on the complex Banach space  $X_+ = C([0, p]; \mathbb{C}^M)$  by*

$$\begin{cases} \mathcal{D}(A_+) = \left\{ \varphi \in C^1([0, p]; \mathbb{C}^M) : \varphi'(0) = \int_0^p d\eta(\theta) \varphi(\theta) \right\}, \\ A_+ \varphi = \varphi' \text{ for } \varphi \in \mathcal{D}(A_+), \end{cases}$$

*has a pure eigenvalue spectrum consisting of the zeros of the determinant of the matrix function*

$$\Delta_+(\lambda) = \lambda I_M - \int_0^p e^{\lambda\theta} d\eta(\theta),$$

*has a compact resolvent, and has all of its eigenvalues concentrated in the half-plane  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \sigma\}$  for some  $\sigma \in \mathbb{R}$ . The operator  $-A_+$  is the infinitesimal generator of a hyperbolic semigroup iff  $\det \Delta_+(\lambda) = 0$  does not have any imaginary zeros.*



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# Index

- $\Pi_\kappa$ -space, 118
- abstract Cauchy problem
  - for exponentially dichotomous operators, 49
  - for semigroup generators, 47
- abstract kinetic equation, 89, 92, 106, 108, 110, 116
- Allan-Bochner-Phillips theorem, 30, 38
- approximant
  - to a triple, 81
- Ascoli-Arzelà theorem, 4
- BGK model equation, 91
- bisemigroup
  - analytic, 16
  - immediately compact, 21
  - immediately norm continuous, 20
  - strongly continuous, 5
  - sun dual, 25
- block operator, 73
- Bochner-Phillips theorem, 30, 38
- Calkin algebra, 99
- Cauchy problem
  - for exponentially dichotomous operators, 49
  - for semigroup generators, 47
- center manifold, 190
- Cesaro mean, 14
- classical solution of abstract Cauchy problem
  - for analytic bisemigroup generators, 53
  - for exponentially dichotomous operators, 50
  - for semigroup generators, 48
- Closed Graph Theorem, 2
- collision operator, 90
- complement
  - $J$ -orthogonal, 119
- conditions (a) and (b), 7
- constituent semigroups, 5
- controllability map
  - of noncausal state linear system, 140
- critical point
  - irregular, 121
  - regular, 121
- cross norm, 37
- delay equation, 189, 206
- differential operator
  - maximal, 126, 135
  - minimal, 127, 135
- dominated convergence theorem, 39
- equation of radiative transfer, 90
- equicontinuity, 4
- equivalence of scalar products, 107, 111, 117
  - counterexample to, 112
- exponential dichotomy, 6
- exponential growth bound
  - of bisemigroup, 6
  - of semigroup, 3
  - pair of, 5
- extended Pritchard-Salamon realization, 155
- extended PS-realization, 155

- factorization
  - left canonical, 63, 143
  - left quasi-canonical, 62, 143
  - right canonical, 63, 143
  - right quasi-canonical, 63, 143
- Fokker-Planck equation
  - for Brownian motion, 115, 130
  - for electron scattering, 116, 130
- function
  - Bochner integrable, 39
  - Pettis integrable, 40
  - simple, 39
  - strongly measurable, 39
  - weakly measurable, 39
- functional differential equation
  - of mixed type, 189
  - retarded, 189, 206
- fundamental decomposition, 118
- fundamental symmetry, 118
- generator
  - of bisemigroup, 6
  - of semigroup, 3
- graph, 2
- Green's function matrix, 191
- Hille's lemma
  - for Bochner integrals, 3, 39
  - for Pettis integrals, 41
- Hille-Yosida-Phillips theorem, 4
- infinitesimal generator
  - of bisemigroup, 6
  - of semigroup, 3
- input
  - of noncausal state linear system, 140
  - of state linear system, 140
- input-output operator
  - of an extended PS-realization, 159
  - of left PS-realization, 150
  - of right PS-realization, 154
- integral
  - Bochner, 39
  - Pettis, 40
- Krein space, 118
- Krein's lemma, 105
- Lagrange identity, 127, 135
- Laplace inversion, 15
- Laplace transform
  - of bisemigroup, 6
  - of constituent semigroups, 6
  - of semigroup, 3
- left Pritchard-Salamon realization, 148
- left PS-realization, 148
- Lyapunov equation, 76, 85
- maximal ideal, 28
- maximal ideal space, 28
- mild solution of abstract Cauchy problem
  - for exponentially dichotomous operators, 49, 50
  - for semigroup generators, 48
- mixed-type functional differential equation, 189
- multiplicative projection, 29
- neutron transport equation, 90
- noncausal state linear system, 140
- observability map
  - of noncausal state linear system, 141
- operator
  - $J$ -adjoint, 119
  - $J$ -positive, 119
  - $J$ -selfadjoint, 119
  - $J$ -unitary, 119
  - bisectorial, 16
  - block, 73
  - closed, 2
  - exponentially dichotomous, 6
  - invertible, 2
  - sectorial, 16
- Orlicz-Pettis theorem, 40
- output
  - of noncausal state linear system, 140

- of state linear system, 140
- part of linear operator, 155
- partition of unity, 133
- Pettis' Measurability Theorem, 39
- Pontryagin space, 118
- prolongation of solution
  - left, 199
  - right, 199
- quasi-derivatives, 134
- realization problem
  - for extended PS-realizations, 179
- reasonable cross norm, 37
- regularity condition, 99
- regularity constraint, 99
- resolvent set, 2
- Riccati equation
  - in a Banach space, 76
  - in a Hilbert space, 80
- right Pritchard-Salamon realization, 151
- right PS-realization, 151
- selfadjoint extension, 127, 135
- semigroup
  - analytic, 16
  - holomorphic, 16
  - hyperbolic, 21
  - immediately compact, 21
  - immediately norm continuous, 20
  - strongly continuous, 3
  - sun dual, 24
- separated boundary conditions, 127
- separating projection, 5
  - Cesaro integral representation of, 15
  - integral representation of, 7
- solution of differential equation
  - of the order  $2n$ , 134
  - of the second order, 126
- spectral function, 120
- spectral radius, 2
- Spectral Theorem
  - for  $J$ -positive operators, 119
- spectrum, 2
- state
  - of noncausal state linear system, 140
  - of state linear system, 140
- state impulse
  - of noncausal state linear system, 140
- state linear system, 140
- streaming operator, 90
- Sturm-Liouville differential
  - expression
    - limit circle, 128
    - limit point, 128
    - of the order  $2n$ , 128
    - of the second order, 126
    - regular, 127, 135
    - singular, 127, 135
- subspace
  - $J$ -indefinite, 119
  - $J$ -negative, 118
  - $J$ -neutral, 118, 119
  - $J$ -nonnegative, 118
  - $J$ -nonpositive, 118
  - $J$ -positive, 118
  - uniformly  $J$ -negative, 119
  - uniformly  $J$ -positive, 119
- subspaces
  - $J$ -orthogonal, 119
- tensor norm
  - injective, 38
  - projective, 38
- transfer function
  - of extended PS-realization, 159
  - of left PS-realization, 150
  - of noncausal state linear system, 141
  - of right PS-realization, 154
- triple, 81
- triple equivalence, 62, 65, 69, 71

- variation of constants formula
  - for exponentially dichotomous operators, 48
  - for semigroup generators, 48
- vector
  - $J$ -negative, 118
  - $J$ -neutral, 118
  - $J$ -nonnegative, 118
  - $J$ -nonpositive, 118
  - $J$ -positive, 118
- weak solution of abstract Cauchy problem
  - for semigroup generators, 48
- weighting pattern
  - of extended PS-realization, 159
  - of left PS-realization, 150
  - of noncausal state linear system, 141
  - of right PS-realization, 154
- Weyl alternative, 128
- Wiener algebra
  - continuous, 29
  - discrete, 28